# Degenerate bifurcation of the rotating patches

Taoufik Hmidi, Joan Mateu

# DEGENERATE BIFURCATION OF THE ROTATING PATCHES

TAOUFIK HMIDI AND JOAN MATEU

Abstract. In this paper we study the existence of doubly-connected rotating patches for Euler equations when the classical non-degeneracy conditions are not satisfied. We prove the bifurcation of the V-states with two-fold symmetry, however for higher $m-$fold symmetry with $m \geq 3$ the bifurcation does not occur. This answers to a problem left open in [19]. Note that, contrary to the known results for simply-connected and doubly-connected cases where the bifurcation is pitchfork, we show that the degenerate bifurcation is actually transcritical. These results are in agreement with the numerical observations recently discussed in [19]. The proofs stem from the local structure of the quadratic form associated to the reduced bifurcation equation.

Contents



## 1. Introduction

In this paper we deal with the vortex motion for incompressible Euler equations in two-dimensional space. The formulation velocity-vorticity is given by the nonlinear transport equation

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \ x \in \mathbb{R}^2, \ t \geq 0, \\ v = \nabla^{\perp} \Delta^{-1} \omega, \\ \omega_{|t=0} = \omega_0, \end{cases} \tag{1}$$

where $v = (v^1, v^2)$ denotes the velocity field and $\omega = \partial_1 v^2 - \partial_2 v^1$ its vorticity. The second equation in (1) is the Biot-Savart law which can be written with a singular operator as follows: By identifying the vector $v = (v_1, v_2)$ with the complex function $v_1 + iv_2$, we may write

$$v(t,z) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\omega(t,\xi)}{\overline{z} - \overline{\xi}} \, dA(\xi), \quad z \in \mathbb{C}, \tag{2}$$

with $dA$ being the planar Lebesgue measure. Global existence of classical solutions was established a long time ago by Wolibner in [35] and follows from the transport structure of the vorticity equation. For a recent account of the theory we refer the reader to [1, 7]. The same result was extended for less regular initial data by Yudovich in [37] who proved that the system (1) admits a unique global solution in the weak sense when the initial vorticity $\omega_0$ is bounded an integrable. This result is of great importance because it enables to deal rigorously with some discontinuous vortices as the vortex patches which are the characteristic function of bounded domains. Therefore, when $\omega_0 = \chi_D$ with $D$ a bounded domain then the solution of (1) preserves this structure for long time and $\omega(t) = \chi_{D_t}$, where $D_t = \psi(t, D)$ being the image of $D$ by the flow. The motion of the patch is governed by the contour dynamics equation which takes the following form : Let $\gamma_t : \mathbb{T} \to \partial D_t$ be the Lagrangian parametrization of the boundary, then

$$\partial_t \gamma_t = -\frac{1}{2\pi} \int_{\partial D_t} \log|\gamma_t - \xi| d\xi.$$

Notice that according to Chemin's result [7], when the initial boundary is of Hölder class $C^{1+\varepsilon}$ with $0 < \varepsilon < 1$, then there is no formation of singularities in finite time and the regularity is therefore propagated for long time. In general, it is not an easy task to tackle the full dynamics of a given vortex patch due the singular nonlinearity governing the motion of its boundary. Nevertheless, we know two explicit examples where the patches perform a steady rotation about their centers without changing shape. The first one is Rankine vortices where the boundary is a circle and this provides the only stationary patches according to Fraenkel's result [13]. As to the second example, it is less obvious and goes back to Kirchhoff [25] who proved that an elliptic patch rotates about its center with the angular velocity $\Omega = ab/(a+b)^2$, where $a$ and $b$ are the semi-axes of the ellipse. For another proof see [1, p.304]. Many years later, several examples of rotating patches, called also $V$-states, were constructed numerically by Deem and Zabusky [9] using contour dynamics algorithm. Note that a rotating patch is merely a patch whose domain $D_t$ rotates uniformly about its center which can be assumed to be the origin. This means that $D_t = e^{it\Omega} D$ and $\Omega \in \mathbb{R}$ being the angular velocity. Later, Burbea provided in [2] an analytical proof and showed the existence of the V-states with $m$-fold symmetry for each integer $m \geq 2$. In this countable family the case $m = 2$ corresponds to Kirchhoff elliptic vortices. Burbea's approach consists in using complex analysis tools combined with the bifurcation theory. It should be noted that from this point of view the rotating patches are arranged in a collection of countable curves bifurcating from Rankine vortices (trivial solution) at the discrete angular velocities set $\left\{\frac{m-1}{2m}, m \geq 2\right\}$. The numerical experiments of the limiting V-states which are the ends of each branch were accomplished in [29, 36] and they reveal interesting behavior; the boundary develops corners at right angles. Recently, Verdera and the authors studied in [17] the regularity of the V-states close to the disc and proved that the boundaries are in fact of class $C^\infty$ and convex. More recently, this result was improved by Castro, Córdoba and Gómez-Serrano in [5] who showed the analyticity of the V-states close to the disc. We mention that similar research has been conducted in the past few years for more singular nonlinear transport equations arising in fluid dynamics as the surface quasi-geostrophic equations or the quasi-geostrophic shallow-water equations; see for instance [4, 5, 11, 15, 16].

It should be noted that the bifurcating curves from the discs are pitchfork and always located in the left-hand side of the point of bifurcation $\frac{m-1}{2m}$. This implies that angular velocities of those V-states are actually contained in the interval $]0, \frac{1}{2}[$. Whether or not other V-states still survive outside this range of $\Omega$ is an interesting problem. The limiting cases $\Omega \in \{0, \frac{1}{2}\}$ are well-understood and it turns out that the discs are the only solutions for the V-states problem. As we have mentioned before, this was proved in the stationary case $\Omega = 0$ by Fraenkel in [13] using the moving plane method. We remark that this method is quite flexible

and was adapted by the first author to $\Omega<0$ but under a convexity restriction, see [20]. With regard to the second endpoint $\Omega=\frac12$ it is somehow more easier than $\Omega=0$ and was solved recently in [20] using the maximum principle for harmonic functions.
The study of the second bifurcation of rotating patches from Kirchhoff ellipses was first examined by Kamm in [22], who gave numerical evidence of the existence of some branches bifurcating from the ellipses, see also [31]. In [27], Luzzatto-Fegiz and Willimason gave more details about the bifurcation diagram of the first curves and where we can also find some nice pictures of the limiting V-states. Very recently, analytical proofs of the existence part of the V-states and their regularity were explored in [5, 21]. Indeed, the authors showed that the bifurcation from the ellipses $\big\{w+Q\overline{w},\, w\in\mathbb{T}\big\}$ parametrized by the associated exterior conformal mapping occurs at the values $Q\in]0,1[$ such that there exists $m\geq 3$ with

$$1+Q^m-\frac{1-Q^2}{2}m=0.$$

The linear and nonlinear stability of long-lived vortex structures is an old subject in fluid dynamics and much research has been carried out since Love's work [26] devoted to Kirchhoff ellipses. The nonlinear stability of the elliptic vortices was explored later in [14, 32]. As to rotating patches of m-fold symmetry, a valuable discussion on the linear stability was done by Burbea and Landau in [3] using complex analysis approach. However the nonlinear stability in a small neighborhood of Rankine vortices was performed by Wan in [33] through some variational arguments. For further numerical discussions, see also [6, 10, 28].
Recently, in [18, 19] close attention was paid to doubly-connected $V$-states taking the form $\chi_D$ with $D=D_1\backslash D_2$ and $D_2\Subset D_1$ being two simply-connected domains. It is apparent from symmetry argument that an annulus is a stationary $V$-state, however and up to our knowledge no other explicit example is known in the literature. Note that in the paper [18] a full characterization of the V-states (with nonzero magnitude in the interior domain) with at least one elliptical interface was performed complementing the results of Flierl and Polvani [12]. As a by-product, the only elliptic doubly-connected V-states are the annuli. The existence of nonradial doubly-connected V-states was studied in [19] in the spirit of Burbea's work but with much more involved calculations. Roughly speaking, starting from the Rankine patch supported by the annulus

$$\mathbb{A}_b\triangleq\{z\in\mathbb{C}, b<|z|<1\}\tag{3}$$

we obtain a collection of countable bifurcating curves from this annulus at some explicit values of the angular velocities. To be more precise, let $m\geq 3$ verifying

$$1+b^m-\frac{1-b^2}{2}m<0,\tag{4}$$

then there exists two curves of doubly-connected rotating patches with $m$-fold symmetry bifurcating from the annulus $\mathbb{A}_b$ at the angular velocities

$$\Omega_m^{\pm}=\frac{1-b^2}{4}\pm\frac{1}{2m}\sqrt{\Delta_m},\quad \Delta_m\triangleq\Big(\frac{m(1-b^2)}{2}-1\Big)^2-b^{2m}.\tag{5}$$

In [19], it is showed that the condition (4) is equivalent to $\Delta_m>0$, in which case the eigenvalues are simple and therefore the transversality assumption required by Crandall-Rabinowitz theorem (CR theorem for abbreviation) is satisfied. The main purpose of the current paper is to investigate the degenerate case corresponding to vanishing discriminant, that is, $\Delta_m=0$. This problem was left open in [19] because the transversality assumption is no longer verified and CR theorem is therefore inefficient. As a matter of fact, the study of the linearized operator is not enough to understand the local structure of the nonlinear problem and answer to the bifurcation problem. Before stating our result we need to introduce the following set $\mathbb{S}$ and describe its structure. This set will be carefully studied in Section 2.2. Let

$$
\begin{aligned}
\mathbb{S} &\triangleq \Big\{(m,b)\in\mathbb{N}^\star\times]0,1[,\ \Delta_m=0\Big\} \\
&= \Big\{(2,b), b\in]0,1[\Big\}\cup\Big\{(m,b_m), m\geq 3\Big\},
\end{aligned} \tag{6}
$$

where for given $m\geq 3$, $b_m$ is the unique solution in $]0,1[$ of the equation

$$1+b^m-\frac{1-b^2}{2}m=0.$$

It is not so hard to show that the sequence $(b_m)_{m\geq 3}$ is strictly increasing and converges to 1. Moreover one can easily check that

$$b_3=\frac{1}{2}\quad \text{and}\quad b_4=\sqrt{\sqrt{2}-1}\approx 0.6435.$$

Notice that from the numerical experiments done in [19] we observe the formation of small loops from the branches emanating from the eigenvalues $\Omega_m^+$ and $\Omega_m^-$ (with $m\geq 3$) when these latter are close enough. It turns out that these loops become very small and shrink to a point which is the trivial solution when the distance between the eigenvalues goes to zero. This suggests that in the degenerate case the bifurcation does not occur. We shall attempt at clarifying this numerical evidence from theoretical standpoint by providing analytical proof based upon the bifurcation theory which however still an efficient tool. We shall also investigate the two-fold symmetry where the situation seems to be completely different from the preceding one and rotating patches continue to exist. We can now formulate our main results.

**Main theorem.** *The following assertions hold true.*

(i) *Let $b\in]0,1[\backslash\big\{b_{2m}, m\geq 2\big\}$, then there exists a curve of 2-fold doubly-connected V-states bifurcating from the annulus $\mathbb{A}_b$ at the angular velocity $\frac{1-b^2}{4}$. Furthermore, the bifurcation is transcritical.*

(ii) *Let $b=b_m$ for some $m\geq 3$. Then there is no bifurcation of $m-$fold V-states from the annulus $\mathbb{A}_b$.*

Before giving some details about the proof some remarks are in order.

**Remark 1.** *As it will be discussed in the proof of the Main theorem postponed in Section 5 we have not only one curve of bifurcation in the two-fold case but actually two curves. However, from geometrical point of view they can be related to each other using a symmetry argument and therefore only one branch is significant.*

**Remark 2.** *The existence of two-fold V-states for $b\in\big\{b_{2n}, n\geq 2\big\}$ remains unresolved. In this case the kernel is two-dimensional and the transversality assumption is not satisfied. We believe that more work is needed in order to explore the bifurcation in this very special case.*

Now we shall sketch the general ideas of the proof of the Main theorem. We start with a suitable formulation of the V-states equations which was set up in the preceding works [2, 18, 19] through the use of the conformal mappings. If we denote by $\phi_j$ the exterior conformal mapping of the domain $D_j$ then from Riemann mapping theorem we get,

$$\phi_j(w)=b_jw+\sum_{n\geq 0}\frac{a_{j,n}}{w^n},\quad a_{j,n}\in\mathbb{R},\quad b_1=1,\ b_2=b.$$

The coefficients are assumed to be real since we shall look for domains which are symmetric with respect to the real axis. Therefore the domain $D$ which is assumed to be smooth, of class $C^1$ at least, rotates steadily with an angular velocity $\Omega$ if and only if

$$\forall\, w\in\mathbb{T},\;\; G_j(\lambda,\phi_1,\phi_2)(w)\triangleq \operatorname{Im}\Big\{\Big((1-\lambda)\overline{\phi_j(w)}+I(\phi_j(w))\Big)\, w\,\phi_j'(w)\Big\}=0,\;\; j=1,2,$$

with

$$\lambda\triangleq 1-2\Omega\quad\text{and}\quad I(z)\triangleq \fint_{\mathbb{T}}\frac{\overline{z}-\overline{\phi_1(\xi)}}{z-\phi_1(\xi)}\phi_1'(\xi)d\xi-\fint_{\mathbb{T}}\frac{\overline{z}-\overline{\phi_2(\xi)}}{z-\phi_2(\xi)}\phi_2'(\xi)d\xi.$$

Here we use the parameter $\lambda$ instead of $\Omega$ as in [18, 19] because we found it more convenient in the final formulae that we shall recall in the next few lines. We remind from [19] the following result : If we denote by $\mathcal{L}_{\lambda,b}$ the linearized operator around the annulus defined by

$$\mathcal{L}_{\lambda,b}h\triangleq \frac{d}{dt}G(\lambda,th)_{|t=0},\quad h=(h_1,h_2).$$

Then it acts as a matrix Fourier multiplier, namely, for

$$h_j(w)=\sum_{n\geq 1}\frac{a_{j,n}}{w^{nm-1}},$$

belonging to some function space that will be precise later we have

$$\mathcal{L}_{\lambda,b}h=\sum_{n\geq 1}M_{nm}(\lambda)\begin{pmatrix}a_{1,n}\\ a_{2,n}\end{pmatrix}e_{nm},\quad e_n=\operatorname{Im}(\overline{w}^n).$$

The matrix $M_n$ is explicitly given for $n\geq 1$ by the formula,

$$M_n(\lambda)\;=\;\begin{pmatrix}n\lambda-1-nb^2 & b^{n+1}\\ -b^n & b\big(n\lambda-n+1\big)\end{pmatrix}.$$

Throughout this paper we shall use the terminology of "nonlinear eigenvalues" or simply eigenvalues to denote the values of $\lambda$ associated to the singular matrix $M_n$ for some $n$. Bearing in mind the definition of $\Delta_n>0$ given in (5) then it is straightforward that when $\Delta_n>0$ the eigenvalues are distinct and they coincide when $\Delta_n$ vanishes. In this latter case $\lambda=\lambda_n=\frac{1+b^2}{2}$ and as it was stressed in [19] the transversality assumption is not satisfied and thereby Crandall-Rabinowitz theorem is no longer useful. In this delicate context the linearized operator is not sufficient to describe the local behavior of the nonlinear functional and therefore one should move to higher expansions of the functional. The bifurcation without transversality is often referred in the literature as *the degenerate bifurcation.* There are some studies devoted to this case; in [23, Section I.16] and [24] Kielhöfer obtained some results on the bifurcation but with some restrictive structure on the nonlinear functional. His approach consists in giving a complete ansatz for the real eigenvalue $\mu(\lambda)$ which is a small perturbation of zero. The bifurcation is then formulated through the platitude degree of $\mu$. In our context, it is not clear whether our singular functional satisfies the conditions listed in [23]. The strategy that we shall follow is slightly bit different from the preceding one; instead of looking for the expansion of the eigenvalue $\mu(\lambda)$ we analyze the expansion at the second order of the reduced bifurcation equation and this will be enough for proving our main result. More precisely, using Lyapunov-Schmidt reduction we transform the infinite-dimensional problem into a two-dimensional one. This is done by using some projections intimately connected to the spectral structure of the linearized operator and the V-states equations reduces in fact to a finite-dimensional equation called *reduced bifurcation equation.* It takes the form,

$$F_2(\lambda,t)=0,\quad F_2:\mathbb{R}\times\mathbb{R}\to\mathbb{R}.$$

In the degenerate case where $(m,b)$ belongs to the set $\mathbb{S}$ defined in (6) the point $(\lambda_m,0)$ is in fact a critical point for $F_2$ and therefore the resolvability of the reduced bifurcation equation should require an expansion of $F_2$ at least at the second order around the critical point. The computations are fairly complicated and long, and ultimately we find that the quadratic form is non-degenerate. Accordingly, we stop the expansion at this order and decide about the occurrence of the bifurcation using perturbative argument. To be more precise, we show the local structure,

$$F_2(\lambda,t)=a_m(\lambda-\lambda_m)^2+b_m t^2+\big((\lambda-\lambda_m)^2+t^2\big)\varepsilon(\lambda,t).$$

with $a_m>0$, $\forall m\geq 2$ and $\lim\limits_{(\lambda,t)\mapsto(\lambda_m,0)}\varepsilon(\lambda,t)=0$. Thus the existence of bifurcating curves is related to the sign of the coefficient $b_m$ which reveals behavior change with respect to the frequency mode $m$. Indeed, when $m\geq 3$ we get always $b_m>0$ asserting that $F_2$ is strictly convex and thereby there is no bifurcation. As to the case $m=2$ we obtain $b_2<0$ implying that the bifurcation equation admits non-trivial solutions parametrized by two different curves emerging from the trivial solution. By symmetry argument of the V-states equations we show that from geometrical standpoint the two curves describe in fact the same V-states meaning that we have only one interesting bifurcating curve.
To end this introduction we shall briefly discuss some numerical experiments concerning the V-states and which are done for us by Francisco de la Hoz. In Figure 1 we plot the bifurcation diagram giving the dependence with respect to the angular velocity $\Omega$ of the first coefficients $a_{1,1}$ and $a_{2,1}$ of the conformal mappings $\phi_1$ and $\phi_2$.

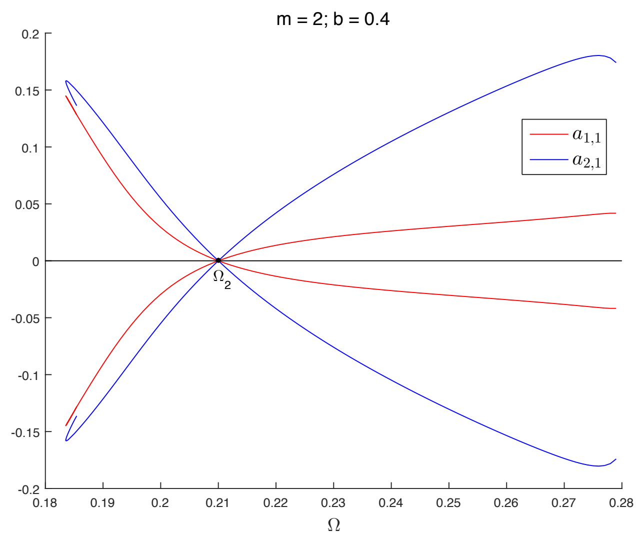


FIGURE 1. Bifurcation diagram.

In Figure 2 we plot the limiting V-states bifurcating from the annulus of small radius $b=0.4$. Note that there are two distinct limiting V-states at the endpoints of the curve of bifurcation and we observe in both cases the formation of two corner singularities at the inner curve, however the outer one remains smooth.
The paper will be organized as follows. In Section 2 we shall discuss the V-states equations and revisit the foundations of the stationary bifurcation theory. In Section 3 we shall deal with the degenerate bifurcation and implement some general computations on the quadratic form associated to the reduced bifurcation equation. In section 4, we implement explicitly all the computations of the preceding section in the special case of the V-states. The proof of the Main theorem will be given in Section 5.

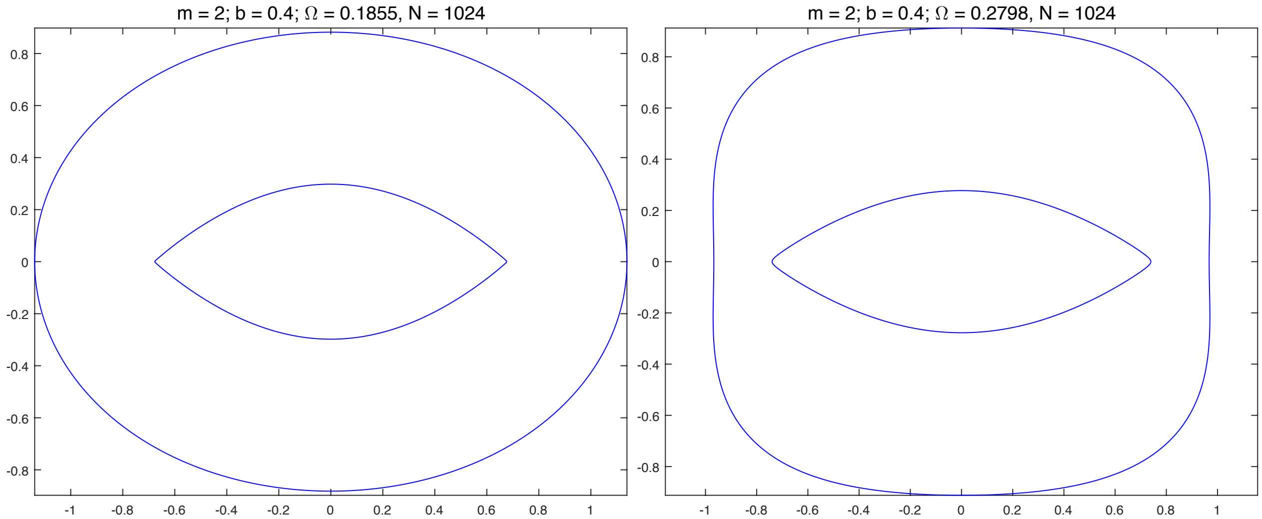


FIGURE 2. Left: the limiting V-state located in the left side of branch of bifurcation. Right: the limiting V-state located in the right side of the same branch.

**Notation.** We need to collect some useful notation that will be frequently used along this paper. We shall use the symbol $\triangleq$ to define an object. Crandall-Rabinowitz theorem is sometimes shorten to CR theorem. The unit disc is denoted by $\mathbb{D}$ and its boundary, the unit circle, by $\mathbb{T}$. For given continuous complex function $f : \mathbb{T} \to \mathbb{C}$, we define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

where $d\tau$ stands for the complex integration.
Let $X$ and $Y$ be two normed spaces. We denote by $\mathcal{L}(X,Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology. We shall denote by $KerT$ and $R(T)$ the null space and the range of $T$, respectively. Finally, if $F$ is a subspace of $Y$, then $Y/F$ denotes the quotient space.

## 2. Preliminaries and background

In this introductory section we shall gather some basic facts on the V-states equations. Firstly, we shall write down the equations using the conformal parametrization which appears to be much more convenient in the computation. Secondly, we will focus on the structure of the linearized operator around the trivial solution and collect some of its relevant spectral properties. This is complemented by Lyapunov-Schmidt reduction which allows us to perform the first step towards the bifurcation and leads to what is referred to as *reduced bifurcation equation.*

2.1. **Boundary equations.** Let $D$ be a doubly-connected domain of the form $D = D_1\backslash D_2$ with $D_2 \Subset D_1$ being two simply-connected domains. Denote by $\Gamma_j$ the boundary of the domain $D_j$. According to [19], the equation of the V-states rotating with the angular velocity $\Omega$ is given by two coupled equations, one per each boundary component $\Gamma_j$.

$$\text{Re}\Big\{\Big((1-\lambda)\overline{z}+I(z)\Big)\,z'\Big\} = 0, \quad \forall\, z \in \Gamma_1 \cup \Gamma_2, \tag{7}$$

with $\lambda = 1 - 2\Omega$ and

$$I(z) = \fint_{\Gamma_1} \frac{\overline{z}-\overline{\xi}}{z-\xi}d\xi - \fint_{\Gamma_2} \frac{\overline{z}-\overline{\xi}}{z-\xi}d\xi.$$

The integrals are defined in the complex sense. Now we shall consider the parametrization of the domains $D_j$ by the exterior conformal mappings: $\phi_j : \mathbb{D}^c \to D_j^c$ satisfying

$$\phi_j(w) = b_j w + \sum_{n\geq 0} \frac{a_{j,n}}{w^n}, \quad a_{j,n} \in \mathbb{N},$$

with $0 < b_j < 1$, $j = 1, 2$ and $b_2 = b < b_1 = 1$. As we shall see later, this parametrization has many advantages especially in the computation of the quadratic form associated to the nonlinear functional through the use of residue theorem. If the boundaries $\Gamma_j = \partial D_j$ are smooth enough then each conformal mapping admits univalent extension to the boundary $\mathbb{T}$, for instance see [30, 34]. The restriction on $\Gamma_j$ is still denoted by $\phi_j$. After a change of variable in the integrals we get

$$I(z) = \fint_{\mathbb{T}} \frac{\overline{z} - \overline{\phi_1(\xi)}}{z - \phi_1(\xi)} \phi_1'(\xi) d\xi - \fint_{\mathbb{T}} \frac{\overline{z} - \overline{\phi_2(\xi)}}{z - \phi_2(\xi)} \phi_2'(\xi) d\xi.$$

Setting $\phi_j = b_j \mathrm{Id} + f_j$ then the equation (7) becomes

$$G(\lambda, f_1, f_2) = 0$$

with $G = (G_1, G_2)$ and

$$G_j(\lambda, f_1, f_2)(w) \triangleq \mathrm{Im}\Big\{ \Big(2\Omega\overline{\phi_j(w)} + I(\phi_j(w))\Big)\, w\, \phi_j'(w) \Big\}. \tag{8}$$

We can easily check that

$$G(\Omega, 0, 0) = 0, \quad \forall \Omega \in \mathbb{R}.$$

This is coherent with the fact that an annulus is a stationary solution for Euler equations and due to the radial symmetry we can say that it rotates with any arbitrary angular velocity.

2.2. **Linearized operator and Lyapunov-Schmidt reduction.** In this section we shall recall some results from [19] concerning the structure of the linearized operator around the trivial solution of the functional $G$ defining the V-states equation (8). The case of double eigenvalues was left open in [19] because we lose the transversality assumption in Crandall-Rabinowitz theorem. Our main focus is to make some preparations in order to understand this degenerate case. The strategy is to come back to the foundation of the bifurcation theory and try to improve these tools. As we shall see, Lyapunov-Schmidt reduction is used to transform the infinite-dimensional problem into a two-dimensional one. Roughly speaking, after some suitable projections we reduce the V-states equation in local coordinates into an equation of the type

$$F_2(\lambda, t) = 0, \quad F_2 : \mathbb{R}^2 \to \mathbb{R}.$$

The points where the transversality is not satisfied correspond merely to critical points for $F_2$. It turns out that the quadratic form associated to $F_2$ giving Taylor expansion at the second order around the critical points is non-degenerate and its signature depends on the frequency $m$. This will be enough to answer to the bifurcation problem. Now we shall introduce the function spaces that will capture the $m-$fold structure of the V-states.

$$X_m = \Big\{ f = (f_1, f_2) \in (C^{1+\alpha}(\mathbb{T}))^2,\ f(w) = \sum_{n=1}^{\infty} A_n \overline{w}^{nm-1},\ A_n \in \mathbb{R}^2 \Big\}. \tag{9}$$

$$Y_m = \Bigg\{ G = (G_1, G_2) \in (C^{\alpha}(\mathbb{T}))^2,\ G = \sum_{n\geq 1} B_n e_{nm},\ B_n \in \mathbb{R}^2 \Bigg\}, e_n(w) = \mathrm{Im}(\overline{w}^n). \tag{10}$$

The proof of the Main theorem consists in solving the V-states equation in a small neighborhood of zero, namely,

$$G(\lambda,f)=0,\quad f=(f_1,f_2)\in B_r^m\times B_r^m\subset X_m, \tag{11}$$

where we denote by $B_r^m$ the ball of radius $r\in(0,1)$ in the space of functions which are one component of an element in $X_m$. The linearized operator around zero is defined by

$$\mathcal{L}_{\lambda,b}h\triangleq D_fG(\lambda,0)h=\frac{d}{dt}G(\lambda,th)_{|t=0}$$

and from [19], we assert that for $h=(h_1,h_2)\in X$ with the expansions

$$h_j(w)=\sum_{n\geq1}\frac{a_{j,n}}{w^{nm-1}},$$

one has

$$\mathcal{L}_{\lambda,b}h=\sum_{n\geq1}M_{nm}(\lambda)\begin{pmatrix}a_{1,n}\\ a_{2,n}\end{pmatrix}e_{nm}. \tag{12}$$

where the matrix $M_n$ is given for $n\geq1$ by

$$M_n(\lambda)\quad=\quad\begin{pmatrix}n\lambda-1-nb^2 & b^{n+1}\\ -b^n & b\big(n\lambda-n+1\big)\end{pmatrix}. \tag{13}$$

The nonlinear eigenvalues are the values of $\lambda$ such that at least one matrix $M_n$ is not invertible. Note the determinant equation is a polynomial of second order and it admits real roots if and only if the discriminant is positive. According to [19, p.28], the roots are explicitly given by

$$\lambda_n^{\pm}=\frac{1+b^2}{2}\pm\frac{1}{n}\sqrt{\Delta_n},$$

with

$$\Delta_n=b^{2n}-\left(\frac{1-b^2}{2}n-1\right)^2\geq0. \tag{14}$$

Bifurcation study when the roots are distinct, that is $\Delta_n>0$, was studied in [19]. All the assumptions of Crandall-Rabinowitz theorem are satisfied and we have two distinct branches of solutions with $n-$fold symmetry. However, when $\Delta_n=0$ for some $n\geq2$ the transversality assumption is not satisfied and therefore CR theorem ceases to give a profitable result. In Figure 3, we draw for each $n\geq2$ the curve $\mathcal{C}_n$ which is the union of the curves $b\in[0,b_n^\star]\mapsto\lambda_n^{\pm}$. Note that for $n\geq3$, the curve associated to the sign $+$ is lying above and communicates with the curve associated to the sign $-$ at exactly the turning point whose abscissa is $b_n^\star$. This latter number giving the lifespan of the curves coincides with $b_n$ defined below in (15). However for $n=2$, $\Delta_2$ vanishes for all the values of $b$ and therefore $\lambda_2^+=\lambda_2^-=\frac{1+b^2}{2}$. Consequently this eigenvalue is strictly increasing with respect to $b$ and it cuts the curve $\mathcal{C}_n$ for each $n\geq3$ at the turning point (with abscissa $b_n$). This illustrates the special dynamics of $\lambda_2$ compared to higher ones and provides somehow insight into why the bifurcation occurs only for two-fold. As we have already mentioned in the Introduction, the main target of this paper is to focus on this degenerate case where $\Delta_m=0$ for some $m\in\mathbb{N}^\star$ and thus $\lambda_n^+=\lambda_n^-=\frac{1+b^2}{2}$. The first step to settle this case is to understand the structure of the following set

$$\mathbb{S}\triangleq\Big\{(m,b)\in\mathbb{N}^\star\times]0,1[,\ \Delta_m=0\Big\}.$$

It is plain that no solution is associated to $m=1$. However for $m=2$, any $b\in]0,1[$ gives rise to an element of $\mathbb{S}$. As to the values $m\geq3$ we can easily check that the map

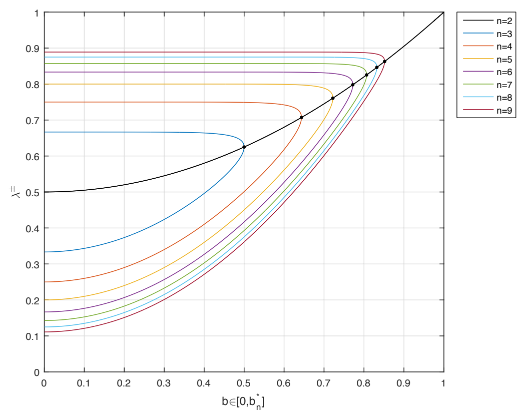


FIGURE 3. The curves of $\mathcal{C}_n$ giving $\lambda_n^{\pm}$ in function of $b$.

$b \mapsto 1 + b^m - \frac{1-b^2}{2}m$ is strictly increasing and therefore it admits only one solution denoted by $b_m$, that is,

$$\frac{1-b_m^2}{2}m - 1 = b_m^m. \tag{15}$$

It is not difficult to check that this sequence is strictly increasing and converges to 1. Hence we deduce the decomposition,

$$\mathbb{S} = \Big\{(2,b), b \in ]0,1[\Big\} \cup \Big\{(m,b_m), m \geq 3\Big\}.$$

For a future use we need to make the notation,

$$\lambda_m = \begin{cases} \frac{1+b^2}{2}, & \text{if} \quad m = 2,\, b \in ]0,1[ \\ \frac{1+b_m^2}{2}, & \text{if} \quad m \geq 3. \end{cases} \tag{16}$$

Now we wish to conduct a spectral study for the matrix $M_m(\lambda_m)$ that will be used later to deduce some spectral properties of the linearized operator $\mathcal{L}_{\lambda,b}$. For $m = 2$ direct computations yield

$$M_2(\lambda_2) = b^2 \begin{pmatrix} -1 & b \\ -1 & b \end{pmatrix}.$$

However, for $m \geq 3$ we can easily check that

$$M_m(\lambda_m) = b^m \begin{pmatrix} 1 & b \\ -1 & -b \end{pmatrix}.$$

We may unify the structure of these matrices as follows,

$$M_m(\lambda_m) = b^m \begin{pmatrix} -\varepsilon & b \\ -1 & \varepsilon\, b \end{pmatrix}, \quad \varepsilon = \begin{cases} 1, & \text{if} \quad m = 2 \\ -1, & \text{if} \quad m \geq 3. \end{cases} \tag{17}$$

To lighten the notation we prefer dropping the subscript on $b_m$ which is noted simply by $b$. We can show without any difficulty that the kernel of $M_m(\lambda_m)$ is one-dimensional and generated by the vector $\begin{pmatrix} \varepsilon b \\ 1 \end{pmatrix}$. Now take $m \geq 2, b = b_m$ and $\lambda = \lambda_m$ then the kernel of the operator $\mathcal{L}_{\lambda_m,b}$ defined by (12) is one-dimensional in $X_m$ and generated by

$$w \in \mathbb{T} \mapsto v_m(w) = \begin{pmatrix} \varepsilon b \\ 1 \end{pmatrix} \overline{w}^{m-1}. \tag{18}$$

However for $m = 2$ and $b \notin \{b_m, m \geq 3\}$ the kernel of $\mathcal{L}_{\lambda_2,b}$ is also one-dimensional in $X_2$ and generated by the same vector (18) with $\varepsilon = 1$. As to the case $m = 2$ and $b = b_n$ for some $n \geq 3$, then the kernel is one-dimensional in $X_2$ if $n$ is odd (since the vector $v_n \notin X_2$) but two-dimensional if $n$ is even. To formulate in a compact way the one-dimensional case we shall introduce the set

$$\widehat{\mathbb{S}} \triangleq \mathbb{S} \backslash \{(2, b_{2m}), m \geq 2\}. \tag{19}$$

Therefore if $(m, b) \in \widehat{\mathbb{S}}$ then the kernel of $\mathcal{L}_{\lambda_m,b}$ is one-dimensional in $X_m$ and generated by (18). In conclusion, the foregoing results can be summarized in the next proposition as follows.

**Proposition 1.** *Let $(m, b) \in \mathbb{S}$ then the following assertions hold true.*

(i) *If $(m, b) \in \widehat{\mathbb{S}}$ then the kernel of $\mathcal{L}_{\lambda_m,b}$ is one-dimensional in $X_m$ and generated by the vector defined in (18).*

(ii) *If $m = 2$ and $b = b_{2n}$ for some $n \geq 2$ then the kernel of $\mathcal{L}_{\lambda_2,b}$ is two-dimensional in $X_2$.*

Our next task is to implement Lyapunov-Schmidt reduction used as a basic tool in the bifurcation theory. One could revisit this procedure in a general setup but for the sake of simplicity we prefer limit the description to our context of the V-states. Let $(m, b) \in \widehat{\mathbb{S}}$ and denote by $\mathcal{X}_m$ any complement of the kernel $\langle v_m \rangle$ in $X_m$. From now on we shall work with the following candidate :

$$h \in \mathcal{X}_m \Longleftrightarrow h \in (C^{1+\alpha}(\mathbb{T}))^2,\ h(w) = \sum_{n \geq 2} A_n \overline{w}^{nm-1} + \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{w}^{m-1}, A_n \in \mathbb{R}^2, \alpha \in \mathbb{R}. \tag{20}$$

It is not hard to check that this sub-space is complete and

$$X_m = \langle v_m \rangle \oplus \mathcal{X}_m.$$

Now denote by $\mathcal{Y}_m$ the range of $D_f G(\lambda_m, 0)$ in $Y_m$ which is explicitly described by :

$$k \in \mathcal{Y}_m \Longleftrightarrow k \in (C^{\alpha}(\mathbb{T}))^2,\ \ k(w) = \sum_{n \geq 2} B_n e_{nm} + \beta \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} e_m, \quad B_n \in \mathbb{R}^2,\ \beta \in \mathbb{R}. \tag{21}$$

Then we observe that it is of co-dimension one and admits as a complement the line $\langle \mathbb{W}_m \rangle$ generated by the vector

$$\begin{aligned} \mathbb{W}_m &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{\varepsilon}{\sqrt{2}} \end{pmatrix} e_m \\ &\triangleq \widehat{\mathbb{W}}\, e_m. \end{aligned}$$

Thus we may write

$$Y_m = \langle \mathbb{W}_m \rangle \oplus \mathcal{Y}_m.$$

Consider two projections $P : X_m \mapsto \langle v_m \rangle$ and $Q : Y_m \mapsto \langle \mathbb{W}_m \rangle$. As we shall see the structure of $P$ does not matter in the computations and therefore there is no need to work with an explicit element. However for the projection $Q$ we need to get an explicit one. This will be given by the "orthogonal projection", that is, for $h \in Y_m$ defined in (10) we set

$$h(w) = \sum_{n\geq 1} B_n e_{nm}, \quad Qh(w) = \big\langle B_1, \widehat{\mathbb{W}} \big\rangle \mathbb{W}_m \tag{22}$$

and $\langle , \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^2$. By virtue of the definition we observe that

$$Q\partial_f G(\lambda_m, 0) = 0. \tag{23}$$

For $f \in X_m$, we define the components,

$$g = Pf \quad \text{and} \quad k = (\mathrm{Id} - P)f.$$

Then the V-states equation (11) is equivalent to the system

$$F_1(\lambda, g, k) \triangleq (\mathrm{Id} - Q)G(\lambda, g + k) = 0 \quad \text{and} \quad QG(\lambda, g + k) = 0.$$

The function $F_1 : \mathbb{R} \times \langle v_m \rangle \times \mathcal{X}_m \to \mathcal{Y}_m$ is well-defined and smooth. The regularity follows from the study done in [19]. Moreover, it is not difficult to check that

$$D_k F_1(\lambda_m, 0, 0) = (\mathrm{Id} - Q)\partial_f G(\lambda_m, 0). \tag{24}$$

Using (23) we deduce that

$$D_k F_1(\lambda_m, 0, 0) = \partial_f G(\lambda_m, 0) : \mathcal{X}_m \to \mathcal{Y}_m,$$

which is invertible and by (17) we can explicitly get its inverse :

$$\partial_f G(\lambda_m, 0)h = k \Longleftrightarrow \forall n \geq 2, A_n = M_{nm}^{-1} B_n \quad \text{and} \quad \alpha = -\frac{\beta}{b^m}\cdot \tag{25}$$

By the Implicit Function Theorem the solutions of the equation $F_1(\lambda_m, g, k) = 0$ are described near the point $(\lambda_m, 0)$ by the parametrization $k = \varphi(\lambda, g)$ with

$$\varphi : \mathbb{R} \times \langle v_m \rangle \to \mathcal{X}_m.$$

Therefore the resolution of V-states equation close to $(\lambda_m, 0)$ is equivalent to

$$QG\big(\lambda, tv_m + \varphi(\lambda, tv_m)\big) = 0.$$

From the assumption of CR theorem $G(\lambda, 0) = 0$, $\forall \lambda$, one can deduce that

$$\varphi(\lambda, 0) = 0, \quad \forall \lambda \quad \text{close to} \quad \lambda_m. \tag{26}$$

Using Taylor formula on the variable $t$ in order to get rid of the trivial solution corresponding to $t = 0$, then nontrivial solutions to (11) are locally described close to $(\lambda_m, 0)$ by the equation

$$F_2(\lambda, t) \triangleq \int_0^1 Q\partial_f G\big(\lambda, stv_m + \varphi(\lambda, stv_m)\big)\big(v_m + \partial_g \varphi(\lambda, stv_m)v_m\big)ds = 0. \tag{27}$$

This equation is called along this paper bifurcation equation. Moreover, using the relation (23) we find

$$F_2(\lambda_m, 0) = 0.$$

## 3. Degenerate bifurcation

In this section we intend to compute the Jacobian and the Hessian of $F_2$ in a general framework. This allows to go beyond Crandall-Rabinowitz theorem and give profitable information when the transversality condition is not satisfied. To be precise about the terminology, we talk about *degenerate bifurcation* when the bifurcation occurs without transversality assumption. First we recall Crandall-Rabinowitz theorem [8] which throughout this paper is often denoted by CR theorem.

**Theorem 1.** *Let $X, Y$ be two Banach spaces, $V$ a neighborhood of $0$ in $X$ and let $F : \mathbb{R} \times V \to Y$ with the following properties:*

(i) $F(\lambda, 0) = 0$ *for any* $\lambda \in \mathbb{R}$.

(ii) *The partial derivatives $F_\lambda$, $F_x$ and $F_{\lambda x}$ exist and are continuous.*

(iii) *The spaces Ker $\mathcal{L}_0$ and $Y/R(\mathcal{L}_0)$ are one-dimensional.*

(iv) *Transversality assumption: $\partial_\lambda \partial_x F(0,0)x_0 \notin R(\mathcal{L}_0)$, where*

$$Ker\ \mathcal{L}_0 = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

*If $Z$ is any complement of Ker $\mathcal{L}_0$ in $X$, then there is a neighborhood $U$ of $(0,0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and continuous functions $\varphi : (-a, a) \to \mathbb{R}$, $\psi : (-a, a) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$ and*

$$F^{-1}(0) \cap U = \Big\{\big(\varphi(\xi), \xi x_0 + \xi\psi(\xi)\big)\,;\, |\xi| < a\Big\} \cup \Big\{(\lambda, 0)\,;\, (\lambda, 0) \in U\Big\}.$$

We adopt in this section the same notation and definitions of the preceding section. Notice that CR theorem is an immediate consequence of the implicit function theorem applied to the function $F_2$ introduced in (27). Indeed, by virtue of Proposition 2-(1) we have

$$\partial_\lambda F_2(\lambda_m, 0) = Q\partial_\lambda \partial_f G(\lambda_m, 0)v_m.$$

It is apparent that the transversality assumption is equivalent to saying $\partial_\lambda F_2(\lambda_m, 0) \neq 0$ and thus we can apply the implicit function theorem. Unfortunately, this condition is no longer satisfied under the conditions of the Main theorem and we should henceforth work with the following assumption :

$$Q\partial_\lambda \partial_f G(\lambda_m, 0)v_m = 0. \tag{28}$$

This means that $\partial_\lambda F_2(\lambda_m, 0) = 0$ and we shall see later in Proposition 3 that $\partial_t F_2(\lambda_m, 0)$ is also vanishing. The existence of nontrivial solutions to $F_2$ will be answered through the study of its Taylor expansion at the second order. In the next lemma we shall give some relations on the first derivatives of the function $\varphi$.

**Lemma 1.** *Assume that the function $G$ satisfies the conditions $(i)-(ii)-(iii)$ of CR theorem. Then*

$$\partial_\lambda \varphi(\lambda_m, 0) = \partial_g \varphi(\lambda_m, 0) = 0.$$

*In addition if $G$ is $C^2$ and satisfies $\partial_{\lambda\lambda} G(\lambda_m, 0) = 0$ then*

$$\partial_{\lambda\lambda} \varphi(\lambda_m, 0) = 0.$$

*Proof.* The first result follows easily from (26). Concerning the second one, we rewrite the equation of $F_1$ previously introduced in Section 2.2 in a neighborhood of $(\lambda_m, 0)$ as follows,

$$(\mathrm{Id} - Q)G\big(\lambda, tv_m + \varphi(\lambda, tv_m)\big) = 0. \tag{29}$$

Differentiating this equation with respect to $t$ we get

$$(\mathrm{Id} - Q)\partial_f G\big(\lambda_m, 0\big)\big(v_m + \partial_g \varphi(\lambda_m, 0)v_m\big) = 0.$$

From (23) and since $v_m \in \text{Ker } \partial_f G\big(\lambda_m, 0\big)$ we may obtain

$$\partial_f G\big(\lambda_m, 0\big)\big(\partial_g\varphi(\lambda_m, 0)v_m\big) = 0.$$

As by definition $(\partial_g\varphi(\lambda_m, 0)v_m \in \mathcal{X}_m$ which a complement of the kernel of $\partial_f G\big(\lambda_m, 0\big)$ then we get

$$\partial_g\varphi(\lambda_m, 0)v_m = 0.$$

It remains to check the last identity. Differentiating the equation (29) twice with respect to $\lambda$ we get

$$\begin{aligned} 0 &= (\text{Id} - Q)\partial_{ff}G\big(\lambda_m, 0\big)\Big[\partial_\lambda\varphi(\lambda_m, 0), \partial_\lambda\varphi(\lambda_m, 0)\Big] \\ &+ 2(\text{Id} - Q)\partial_f\partial_\lambda G\big(\lambda_m, 0\big)\partial_\lambda\varphi(\lambda_m, 0) + (\text{Id} - Q)\partial_f G\big(\lambda_m, 0\big)\partial_{\lambda\lambda}\varphi(\lambda_m, 0) \\ &+ (\text{Id} - Q)\partial_{\lambda\lambda}G\big(\lambda_m, 0\big). \end{aligned}$$

In the preceding identity we have used the following notation: for $v, w \in X$, $\partial_{ff}G[v, w]$ denotes the value of the second derivative at the vector $(v, w)$. Using the relations $\partial_\lambda\varphi(\lambda_m, 0) = \partial_g\varphi(\lambda_m, 0) = 0$ combined with the fact $\partial_{\lambda\lambda}G(\lambda_m, 0) = 0$ we get

$$(\text{Id} - Q)\partial_f G\big(\lambda_m, 0\big)\partial_{\lambda\lambda}\varphi(\lambda_m, 0) = 0.$$

By (23) we conclude that $\partial_{\lambda\lambda}\varphi(\lambda_m, 0) \in \text{Ker } \partial_f G\big(\lambda_m, 0\big)$ and we know from the image of $\varphi$ that this vector belongs also to the vector space $\mathcal{X}_m$ which is a complement of the kernel. Therefore we deduce that

$$\partial_{\lambda\lambda}\varphi(\lambda_m, 0) = 0.$$

This achieves the proof. $\square$

Next we shall be concerned with some basic formulae for the Jacobian and the Hessian of the function $F_2$ introduced in (27).

**Proposition 2.** *Assume that $G$ is sufficiently smooth ( of class $C^3$) and satisfies the conditions $(i) - (ii) - (iii)$ of CR theorem. Then the following assertions hold true.*

(i) *First derivatives:*

$$\partial_\lambda F_2(\lambda_m, 0) = Q\partial_\lambda\partial_f G\big(\lambda_m, 0\big)v_m.$$

$$\begin{aligned} 2\partial_t F_2(\lambda_m, 0) &= Q\partial_{ff}G(\lambda_m, 0)\big[v_m, v_m\big] \\ &= \frac{d^2}{dt^2}QG(\lambda_m, tv_m)_{|t=0}. \end{aligned}$$

(ii) *Formula for $\partial_{tt}F_2(\lambda_m, 0)$.*

$$\partial_{tt}F_2(\lambda_m, 0) = \frac{1}{3}\frac{d^3}{dt^3}QG(\lambda_m, tv_m)_{|t=0} + Q\partial_{ff}G(\lambda_m, 0)\big[v_m, \widetilde{v}_m\big]$$

*with $\widetilde{v}_m = \frac{d^2}{dt^2}\varphi(\lambda_m, tv_m)_{|t=0}$ and it is given by*

$$\widetilde{v}_m = -\Big[\partial_f G(\lambda_m, 0)\Big]^{-1}\frac{d^2}{dt^2}(\text{Id} - Q)G(\lambda_m, tv_m)_{|t=0}$$

$$Q\partial_{ff}G(\lambda_m, 0)\big[v_m, \widetilde{v}_m\big] = Q\partial_t\partial_s G(\lambda_m, tv_m + s\widetilde{v}_m)_{|t=0,s=0}$$

(iii) *Formula for $\partial_t\partial_\lambda F_2(\lambda_m, 0)$.*

$$\begin{aligned} \partial_t\partial_\lambda F_2(\lambda_m, 0) &= \frac{1}{2}Q\partial_\lambda\partial_{ff}G(\lambda_m, 0)[v_m, v_m] + Q\partial_{ff}G(\lambda_m, 0)\big[\partial_\lambda\partial_g\varphi(\lambda_m, 0)v_m, v_m\big] \\ &+ \frac{1}{2}Q\partial_\lambda\partial_f G(\lambda_m, 0)\widetilde{v}_m \end{aligned}$$

*and*

$$\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m=-\Big[\partial_f G(\lambda_m,0)\Big]^{-1}(Id-Q)\partial_\lambda\partial_f G(\lambda_m,0)v_m.$$

(iv) *Formula for $\partial_{\lambda\lambda}F_2(\lambda_m,0)$. Assume in addition that $\partial_{\lambda\lambda}G(\lambda_m,0)=0$, then*

$$\partial_{\lambda\lambda}F_2(\lambda_m,0)=-2Q\partial_\lambda\partial_f G(\lambda_m,0)\Big[\partial_f G(\lambda_m,0)\Big]^{-1}(\mathrm{Id}-Q)\partial_\lambda\partial_f G(\lambda_m,0)v_m.$$

**Remark 3.** *The loss of transversality which has been expressed by the identity (28) should be combined with the formulae of Proposition 2 in order to obtain the desired result in the degenerate bifurcation.*

*Proof.* (**1**) Differentiating $F_2$ given by (27) with respect to $\lambda$ gives

$$\begin{aligned}\partial_\lambda F_2\big(\lambda_m,0\big) &= Q\partial_{ff}G\big(\lambda_m,0\big)\big[\partial_\lambda\varphi(\lambda_m,0),\big(v_m+\partial_g\varphi(\lambda_m,0)v_m\big)\big]\\ &+ Q\partial_\lambda\partial_f G\big(\lambda_m,0\big)\big(v_m+\partial_g\varphi(\lambda_m,0)v_m\big)\\ &+ Q\partial_f G\big(\lambda_m,0\big)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m.\end{aligned}$$

Using Lemma 1 we get

$$\partial_\lambda F_2\big(\lambda_m,0\big)=Q\partial_\lambda\partial_f G\big(\lambda_m,0\big)v_m+Q\partial_f G\big(\lambda_m,0\big)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m.$$

Combined with (23) we entail that

$$\partial_\lambda F_2\big(\lambda_m,0\big)=Q\partial_\lambda\partial_f G\big(\lambda_m,0\big)v_m.$$

For the second identity we differentiate (27) with respect to $t$, then we get after straightforward computations

$$\begin{aligned}\partial_t F_2(\lambda_m,t) &= \int_0^1 sQ\partial_{ff}G\big(\lambda_m,stv_m+\varphi(\lambda_m,stv_m)\big)\big[z_m(t,s),z_m(t,s)\big]ds\\ &+ \int_0^1 sQ\partial_f G\big(\lambda_m,stv_m+\varphi(\lambda_m,stv_m)\big)\partial_{gg}\varphi(\lambda_m,stv_m)\big[v_m,v_m\big]ds\end{aligned}\tag{30}$$

with the notation

$$z_m(t,s)\triangleq v_m+(\partial_g\varphi)(\lambda_m,stv_m)v_m.$$

Using Lemma 1 we deduce that

$$2\partial_t F_2\big(\lambda_m,0\big)=Q\partial_{ff}G\big(\lambda_m,0\big)\big[v_m,v_m\big]+Q\partial_f G\big(\lambda_m,0\big)\partial_{gg}\varphi(\lambda_m,0)\big[v_m,v_m\big]\big).$$

Thus we obtain by virtue of (23)

$$2\partial_t F_2\big(\lambda_m,0\big)=Q\partial_{ff}G\big(\lambda_m,0\big)\big[v_m,v_m\big].$$

Now it is easy to check by differentiating the function $t\mapsto QG(\lambda_m,tv_m)$ twice at $t=0$ that

$$2\partial_t F_2\big(\lambda_m,t\big)=\frac{d^2}{dt^2}QG\big(\lambda_m,tv_m\big)_{|t=0}.$$

(**2**) Differentiating (30) with respect to $t$ we get

$$\begin{aligned}\partial_{tt}F_2(\lambda_m,t) &= \int_0^1 s^2Q\partial_{fff}G\big(\lambda_m,stv_m+\varphi(\lambda_m,stv_m)\big)\big[z_m(t,s),z_m(t,s),z_m(t,s)\big]ds\\ &+ 3\int_0^1 s^2Q\partial_{ff}G\big(\lambda_m,stv_m+\varphi(\lambda_m,stv_m)\big)\Big[z_m(t,s),\partial_{gg}\varphi(\lambda_m,stv_m)\big[v_m,v_m\big]\Big]ds\\ &+ \int_0^1 s^2Q\partial_f G\big(\lambda_m,stv_m+\varphi(\lambda_m,stv_m)\big)\partial_{ggg}\varphi(\lambda_m,stv_m)\big[v_m,v_m,v_m\big]ds.\end{aligned}$$

Using Lemma 1, (23) and (26) we get

$$\begin{aligned}\partial_{tt}F_2(\lambda_m,0) &= \frac{1}{3}\partial_{fff}QG\big(\lambda_m,0\big)\big[v_m,v_m,v_m\big]\\ &+ Q\partial_{ff}G\big(\lambda_m,0\big)\Big[v_m,\partial_{gg}\varphi(\lambda_m,0)\big[v_m,v_m\big]\Big].\end{aligned}$$

By setting $\widetilde{v}_m \triangleq \partial_{gg}\varphi(\lambda_m,0)\big[v_m,v_m\big]$ the preceding identity can be written in the form

$$\begin{aligned}\partial_{tt}F_2(\lambda_m,0) &= \frac{1}{3}\frac{d^3}{dt^3}QG\big(\lambda_m,tv_m\big)_{|t=0}+Q\partial_{ff}G\big(\lambda_m,0\big)\big[v_m,\widetilde{v}_m\big]\\ &= \frac{1}{3}\frac{d^3}{dt^3}QG\big(\lambda_m,tv_m\big)_{|t=0}+\partial_t\partial_sQG\big(\lambda_m,tv_m+s\widetilde{v}_m\big)_{|t=0,s=0}.\end{aligned}$$

It remains to compute $\widetilde{v}_m$. Differentiating the equation (29) twice with respect to $t$, we obtain

$$(\mathrm{Id}-Q)\partial_{ff}G(\lambda_m,0)\big[v_m,v_m\big]+(\mathrm{Id}-Q)\partial_fG(\lambda_m,0)\widetilde{v}_m=0.$$

Since $(\mathrm{Id}-Q)\partial_fG(\lambda_m,0):\mathcal{X}_m\to\mathcal{Y}_m$ is invertible then we get in view of (23)

$$\begin{aligned}\widetilde{v}_m &= -\Big[\partial_fG(\lambda_m,0)\Big]^{-1}(\mathrm{Id}-Q)\partial_{ff}G(\lambda_m,0)\big[v_m,v_m\big]\\ &= -\Big[\partial_fG(\lambda_m,0)\Big]^{-1}\frac{d^2}{dt^2}(\mathrm{Id}-Q)G(\lambda_m,tv_m)_{|t=0}\end{aligned}$$

which is the desired identity for $\partial_{tt}F_2(\lambda_m,0)$.

(**3**) To compute $\partial_t\partial_\lambda F_2(\lambda_m,0)$ we proceed as before with direct computations combined with Lemma 1

$$\begin{aligned}\partial_t\partial_\lambda F_2(\lambda_m,0) &= \frac{1}{2}\partial_\lambda\partial_{ff}QG(\lambda_m,0)[v_m,v_m]+\frac{1}{2}Q\partial_\lambda\partial_fG(\lambda_m,0)\partial_{gg}\varphi(\lambda_m,0)[v_m,v_m]\\ &+ Q\partial_{ff}G(\lambda_m,0)\big[v_m,\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m\big]\\ &= \frac{1}{2}\partial_\lambda\partial_{ff}QG(\lambda_m,0)[v_m,v_m]+\frac{1}{2}Q\partial_\lambda\partial_fG(\lambda_m,0)\widetilde{v}_m\\ &+ Q\partial_{ff}G(\lambda_m,0)\big[v_m,\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m\big].\end{aligned}$$

Differentiating (29) respectively with respect to $t$ and $\lambda$ we get

$$(\mathrm{Id}-Q)\partial_\lambda\partial_fG(\lambda_m,0)v_m+(\mathrm{Id}-Q)\partial_fG(\lambda_m,0)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m=0.$$

Since the restricted operator $\partial_fG(\lambda_m,0):\mathcal{X}_m\to\mathcal{Y}_m$ is invertible then

$$\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m=-\Big[\partial_fG(\lambda_m,0)\Big]^{-1}(\mathrm{Id}-Q)\partial_\lambda\partial_fG(\lambda_m,0)v_m. \tag{31}$$

This achieves the proof of the desired identity.

(**4**) Straightforward computations combined with the first part of Lemma 1 yield

$$\begin{aligned}\partial_{\lambda\lambda}F_2(\lambda_m,0) &= Q\partial_{\lambda\lambda}\partial_fG(\lambda_m,0)v_m+2Q\partial_\lambda\partial_fG(\lambda_m,0)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m\\ &+ Q\partial_{ff}G(\lambda_m,0)[\partial_{\lambda\lambda}\varphi(\lambda_m,0),v_m]+Q\partial_fG(\lambda_m,0)\partial_{\lambda\lambda}\partial_g\varphi(\lambda_m,0)v_m.\end{aligned}$$

Using the second part of Lemma 1 where we need the restriction $\partial_{\lambda\lambda}G(\lambda_m,0)=0$ combined with (23) we deduce

$$\partial_{\lambda\lambda}F_2(\lambda_m,0)=2Q\partial_\lambda\partial_fG(\lambda_m,0)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m.$$

It suffices now to use (31) and the desired identity follows.

$\square$

## 4. Jacobian and Hessian computation

As we have already mentioned in Section 2.2 the existence of nontrivial solutions to the equation (11) in a small neighborhood of the point $(\lambda_m, 0)$ defined in (16) is equivalent to solving the finite-dimensional equation (27) :

$$F_2(\lambda, t) = 0, \quad (\lambda, t) \quad \text{close} \quad (\lambda_m, 0).$$

We shall be able to decide from the quadratic form associated to $F_2$ whether or not the function $F_2$ admits nontrivial solutions and prove the Main theorem. We shall establish that the point $(\lambda_m, 0)$ is a critical point for $F_2$ and the Hessian is a diagonal matrix whose eigenvalues change the sign with respect to the frequency $m$. Proposition 2 stresses the complexity of the general formulae for the first and second derivatives and naturally the computations will be very long. Thus we shall take special care in the computations and give the required details. The main result of this section reads as follows.

**Proposition 3.** *Let $(m, b) \in \widehat{\mathbb{S}}$ which was defined in (19). Then the following assertions hold true.*

(i) *First derivatives: we have*

$$\partial_t F_2(\lambda_m, 0) = \partial_\lambda F_2(\lambda_m, 0) = 0.$$

(ii) *Expression of $\partial_{\lambda\lambda} F_2(\lambda_m, 0)$.*

$$\partial_{\lambda\lambda} F_2(\lambda_m, 0) = \sqrt{2} m^2 b^{1-m}\, \mathbb{W}_m.$$

(iii) *Expression of $\partial_t \partial_\lambda F_2(\lambda_m, 0)$.*

$$\partial_t \partial_\lambda F_2(\lambda_m, 0) = 0.$$

(iv) *Expression of $\partial_{tt} F_2(\lambda_m, 0)$.*

$$\partial_{tt} F_2(\lambda_m, 0) = \begin{cases} -\frac{\sqrt{2}(1-b^2)^2}{b}\, \mathbb{W}_2, & \text{if} \quad m = 2, \\ \widehat{\alpha}_m\, \mathbb{W}_m, & \text{if} \quad m \geq 3, \end{cases}$$

*where*

$$\widehat{\alpha}_m = \frac{1}{\sqrt{2}} \Big[ m(m-1) \frac{(1-b^2)^2}{b} + \widehat{\beta}_m \widehat{\gamma}_m \Big]$$

*with*

$$\widehat{\beta}_m = \frac{2m(b^2 + b^m)^2}{(b^m + 1)^2(-b^{2m} + 2b^m + 1)}$$

*and*

$$\widehat{\gamma}_m = (m-2)b + m b^{2m-1} + (4m-6) b^{m+1} + 4(m-1) b^{2m+1} + 2m b^{3m-1}.$$

The proof is quite involved and falls naturally into several steps.

4.1. **Preparation.** The basic tools and notation that we shall deal with in this part were previously introduced in Section 2.2 and for the clarity we refer the reader to this section. Further notation will also be needed :

$$b_1 = 1, b_2 = b, \alpha_1 = \varepsilon b, \alpha_2 = 1. \tag{32}$$

By introducing the functions :

$$\phi_j(t,w) = b_j w + t\alpha_j \overline{w}^{m-1}. \tag{33}$$

one can rewrite the functions of (8) as follows

$$G_j(\lambda_m, tv_m) = \mathrm{Im}\left\{\Big[(1-\lambda_m)\overline{\phi_j(t,w)} + I(\phi_j(w))\Big] w\Big(b_j + t(1-m)\alpha_j\overline{w}^m\Big)\right\}, \tag{34}$$

with

$$I(\phi_j(t,w)) = I_1(\phi_j(t,w)) - I_2(\phi_j(t,w))$$

and

$$I_i(\phi_j(t,w)) = \fint_{\mathbb{T}} \frac{\overline{\phi_j(t,w)} - \overline{\phi_i(t,\tau)}}{\phi_j(t,w) - \phi_i(t,\tau)} \phi_i'(t,\tau)d\tau.$$

According to Proposition 2 the quantities $\frac{d^k}{dt^k}G(\lambda_m, tv_m)_{|t=0}$ for $k = 2, 3$ play central role on the structure of Taylor expansion of $F_2$ at the second order. Their calculations are fairly complicated and as we shall see throughout the next sections we will be able to give explicit expressions.

4.2. **Jacobian computation.** In this section we aim to prove the first point (1) of Proposition 3 which merely says that the point $(\lambda_m, 0)$ is a critical point for the function $F_2$ defined in (27). In other words,

$$\partial_\lambda F_2(\lambda_m, 0) = \partial_t F_2(\lambda_m, 0) = 0.$$

The first one is an immediate consequence of Proposition 2-(1) combined with the loss of transversality written in the form (28). In regard to the second identity we shall use once again Proposition 2-(1)

$$\partial_t F_2(\lambda_m, 0) = \frac{1}{2}\frac{d^2}{dt^2} QG(\lambda_m, tv_m)_{|t=0}.$$

The basic ingredient is the following identity whose explicit formula will be proved and used again later,

$$\begin{aligned} \frac{d^2}{dt^2}G(\lambda_m, tv_m)_{|t=0} &= 2m \begin{pmatrix} \left(b^2 - \varepsilon b^m\right)^2 \\ 0 \end{pmatrix} e_{2m} \\ &\triangleq \begin{pmatrix} \widehat{\alpha} \\ 0 \end{pmatrix} e_{2m}. \end{aligned} \tag{35}$$

Assume for a while this expression, then we get by means of the structure of the projection $Q$ introduced in (22)

$$\frac{d^2}{dt^2} QG(\lambda_m, tv_m)_{|t=0} = 0 \tag{36}$$

and consequently

$$\partial_t F_2(\lambda_m, 0) = 0.$$

Now we shall turn to the proof of (35). Before proving that it is useful to introduce some notation. Set

$$A = b_j w - b_i\tau, \quad B = \alpha_j\overline{w}^{m-1} - \alpha_i\overline{\tau}^{m-1} \quad C = \alpha_i(1-m)\overline{\tau}^m \tag{37}$$

then we may write

$$I_i(\phi_j(t,w)) = \fint_{\mathbb{T}} \frac{\overline{A}+t\overline{B}}{A+tB}\big(b_i+tC\big)d\tau,$$

with $\phi_j$ being the function already defined in (33). It is not so hard to check that

$$\begin{aligned}
\frac{d^2}{dt^2}G_j(\lambda_m, tv_m)_{|t=0} &= \operatorname{Im}\Big\{2(1-\lambda_m)\alpha_j^2(1-m) + b_j w\frac{d^2}{dt^2}I(\phi_j(t))_{|t=0} \\
&+ 2(1-m)\alpha_j\overline{w}^{m-1}\frac{d}{dt}I(\phi_j(t))_{|t=0}\Big\} \\
&= \operatorname{Im}\Big\{b_j w\frac{d^2}{dt^2}I(\phi_j(t))_{|t=0} + 2(1-m)\alpha_j\overline{w}^{m-1}\frac{d}{dt}I(\phi_j(t))_{|t=0}\Big\}. \qquad (38)
\end{aligned}$$

The next task is to compute each term appearing in the right-hand side of the preceding identity.

**Computation of** $\frac{d}{dt}I(\phi_j(t))_{|t=0}$. By the usual rules of the derivation we can check easily that

$$\frac{d}{dt}(I_i(\phi_j(t,\cdot))_{|t=0} = \fint_{\mathbb{T}} \frac{\overline{A}}{A^2}\Big(AC - b_iB\Big)d\tau + b_i\fint_{\mathbb{T}}\frac{\overline{B}}{A}d\tau.$$

It is convenient at this stage to make some scaling arguments which are frequently used later. By homogeneity argument that we can implement by change of variables we find real constants $\mu_{ij}, \gamma_{ij}$ such that

$$\fint_{\mathbb{T}}\frac{\overline{B}}{A}d\tau = \mu_{i,j}w^{m-1}$$

and

$$\fint_{\mathbb{T}} \frac{\overline{A}}{A^2}\Big(AC - b_iB\Big)d\tau = \gamma_{ij}\overline{w}^{m+1},$$

with the expression

$$\gamma_{ij} = \alpha_i(1-m)\fint_{\mathbb{T}}\frac{b_j-b_i\overline{\tau}}{b_j-b_i\tau}\overline{\tau}^m d\tau - b_i\fint_{\mathbb{T}}\frac{b_j-b_i\overline{\tau}}{(b_j-b_i\tau)^2}\big(\alpha_j-\alpha_i\overline{\tau}^{m-1}\big)d\tau.$$

Coming back to the notation (32) we may write

$$\gamma_{12} = \alpha_1(1-m)\fint_{\mathbb{T}}\frac{b-\overline{\tau}}{b-\tau}\overline{\tau}^m d\tau - b_1\fint_{\mathbb{T}}\frac{b-\overline{\tau}}{(b-\tau)^2}\big(\alpha_2-\alpha_1\overline{\tau}^{m-1}\big)d\tau.$$

Since the integrands decay quickly more than $\frac{1}{\tau^2}$ the applying residue theorem at $\infty$ we get,

$$\gamma_{12} = 0. \qquad (39)$$

Again with (32) we write

$$\gamma_{21} = (1-m)\fint_{\mathbb{T}}\frac{1-b\overline{\tau}}{1-b\tau}\overline{\tau}^m d\tau - b\fint_{\mathbb{T}}\frac{1-b\overline{\tau}}{(1-b\tau)^2}\big(\varepsilon b-\overline{\tau}^{m-1}\big)d\tau.$$

Next we shall make use of the following identities whose proofs can be easily obtained through residue theorem : For $m\in\mathbb{N}^\star$

$$\begin{aligned}
\fint_{\mathbb{T}}\frac{\overline{\tau}^m}{1-b\tau}d\tau &= \fint_{\mathbb{T}}\frac{\tau^{m-1}}{\tau-b}d\tau \\
&= b^{m-1} \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}\fint_{\mathbb{T}}\frac{\overline{\tau}^m}{(1-b\tau)^2}d\tau &= \fint_{\mathbb{T}}\frac{\tau^m}{(\tau-b)^2}d\tau\\ &= m\,b^{m-1}.\end{aligned}\tag{41}$$

Consequently we find

$$\begin{aligned}\gamma_{21} &= (1-m)\big(b^{m-1}-b^{m+1}\big)-b\big(-(m-1)b^{m-2}-\varepsilon b^2+mb^m\big)\\ &= \varepsilon b^3-b^{m+1}.\end{aligned}\tag{42}$$

For the diagonal terms $i=j$, we write by the definition

$$\gamma_{ii}=\alpha_i(1-m)\fint_{\mathbb{T}}\frac{1-\overline{\tau}}{1-\tau}\overline{\tau}^m d\tau-\alpha_i\fint_{\mathbb{T}}\frac{1-\overline{\tau}}{(1-\tau)^2}\big(1-\overline{\tau}^{m-1}\big)d\tau.$$

Observe that the integrands are extended holomorphically to $\mathbb{C}^\star$ and therefore by using residue theorem at $\infty$ we get

$$\gamma_{ii}=0.\tag{43}$$

Putting together (39),(42) and (43) we infer

$$\begin{aligned}\frac{d}{dt}I(\phi_1(t,w))_{|t=0} &= \frac{d}{dt}I_1(\phi_1(t,w))_{|t=0}-\frac{d}{dt}I_2(\phi_1(t,w))_{|t=0}\\ &= \big(\gamma_{11}-\gamma_{21}\big)\overline{w}^{m+1}+(\mu_{11}-b\mu_{21})w^{m-1}\\ &= (b^{m+1}-\varepsilon b^3)\overline{w}^{m+1}+(\mu_{11}-b\mu_{21})w^{m-1}.\end{aligned}\tag{44}$$

Proceeding in a similar way to (44) we obtain

$$\begin{aligned}\frac{d}{dt}I(\phi_2(t,w))_{|t=0} &= \frac{d}{dt}I_1(\phi_2(t,w))_{|t=0}-\frac{d}{dt}I_2(\phi_2(t,w))_{|t=0}\\ &= \big(\gamma_{12}-\gamma_{22}\big)\overline{w}^{m+1}+(\mu_{12}-b\mu_{22})w^{m-1}\\ &= (\mu_{12}-b\mu_{22})w^{m-1}.\end{aligned}\tag{45}$$

**Computation of $\frac{d^2}{dt^2}I(\phi_j(t))_{|t=0}$.** Straightforward computations yield

$$\frac{d^2}{dt^2}(I_i(\phi_j(t,\cdot))_{|t=0})=2\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A^3}\Big(AC-b_iB\Big)d\tau.$$

By scaling one finds real constants $\widehat{\mu}_{i,j}$ and $\eta_{i,j}$ such that

$$\fint_{\mathbb{T}}\frac{\overline{B}}{A^2}\Big(AC-b_iB\Big)d\tau=\widehat{\mu}_{i,j}\overline{w},\quad \fint_{\mathbb{T}}\frac{\overline{A}B}{A^3}\Big(b_iB-AC\Big)d\tau=\eta_{i,j}\overline{w}^{2m+1},$$

with

$$\widehat{\mu}_{i,j}=\fint_{\mathbb{T}}\frac{\alpha_j-\alpha_i\tau^{m-1}}{(b_j-b_i\tau)^2}\Big(\alpha_i(1-m)(b_j-b_i\tau)\overline{\tau}^m-b_i(\alpha_j-\alpha_i\overline{\tau}^{m-1})\Big)d\tau.$$

Hence we deduce

$$\frac{d^2}{dt^2}(I_i(\phi_j(t,\cdot))_{|t=0}=2\widehat{\mu}_{i,j}\overline{w}+2\eta_{i,j}\overline{w}^{2m+1}.$$

Thus we obtain successively

$$\frac{d^2}{dt^2}(I(\phi_1(t,\cdot))_{|t=0}=2(\widehat{\mu}_{1,1}-\widehat{\mu}_{2,1})\overline{w}+2(\eta_{1,1}-\eta_{2,1})\overline{w}^{2m+1}$$

$$\frac{d^2}{dt^2}(I(\phi_2(t,\cdot))_{|t=0}=2(\widehat{\mu}_{1,2}-\widehat{\mu}_{2,2})\overline{w}+2(\eta_{1,2}-\eta_{2,2})\overline{w}^{2m+1}.$$

We are left with the task of determining all the involved coefficients. We start with computing $\widehat{\mu}_{i,i}$ which follows easily from residue theorem at $\infty$

$$\begin{aligned}\widehat{\mu}_{i,i} &= \frac{\alpha_i^2}{b_i}\fint_{\mathbb{T}}\frac{1-\tau^{m-1}}{(1-\tau)^2}\Big((1-m)(1-\tau)\overline{\tau}^m-(1-\overline{\tau}^{m-1})\Big)d\tau\\ &= -\frac{\alpha_i^2}{b_i}\fint_{\mathbb{T}}\frac{1-\tau^{m-1}}{(1-\tau)^2}d\tau\\ &= (m-1)\frac{\alpha_i^2}{b_i}.\end{aligned}$$

This implies

$$\widehat{\mu}_{1,1}=(m-1)b^2,\quad \widehat{\mu}_{2,2}=(m-1)\frac{1}{b}\cdot \tag{46}$$

For the coefficient $\widehat{\mu}_{1,2}$ we may write in view of (41)

$$\begin{aligned}\widehat{\mu}_{1,2} &= \fint_{\mathbb{T}}\frac{1-\varepsilon b\tau^{m-1}}{(b-\tau)^2}\Big(\varepsilon b(1-m)(b-\tau)\overline{\tau}^m-(1-\varepsilon b\overline{\tau}^{m-1})\Big)d\tau\\ &= -\fint_{\mathbb{T}}\frac{1-\varepsilon b\tau^{m-1}}{(b-\tau)^2}d\tau\\ &= \varepsilon(m-1)b^{m-1}. \qquad (47)\end{aligned}$$

As to the coefficient $\widehat{\mu}_{2,1}$ we apply once again the residue theorem,

$$\begin{aligned}\widehat{\mu}_{2,1} &= \fint_{\mathbb{T}}\frac{\varepsilon b-\tau^{m-1}}{(1-b\tau)^2}\Big((1-m)(1-b\tau)\overline{\tau}^m-b(\varepsilon b-\overline{\tau}^{m-1})\Big)d\tau\\ &= \fint_{\mathbb{T}}\frac{\varepsilon b-\tau^{m-1}}{(1-b\tau)^2}\Big((1-m)\overline{\tau}^m+bm\overline{\tau}^{m-1}\Big)d\tau\\ &= \fint_{\mathbb{T}}\frac{\varepsilon b(1-m)\overline{\tau}^m+\varepsilon b^2m\overline{\tau}^{m-1}+(m-1)\overline{\tau}}{(1-b\tau)^2}d\tau.\end{aligned}$$

Using (41) and after some cancellations we get

$$\widehat{\mu}_{2,1}=(m-1). \tag{48}$$

Putting together (46),(47) and (48) we infer

$$\begin{aligned}\frac{d^2}{dt^2}(I(\phi_1(t,\cdot))_{|t=0} &= 2(m-1)\big(b^2-1\big)\overline{w}+2(\eta_{1,1}-\eta_{2,1})\overline{w}^{2m+1},\\ \frac{d^2}{dt^2}(I(\phi_2(t,\cdot))_{|t=0} &= 2(m-1)\big(\varepsilon b^{m-1}-b^{-1}\big)\overline{w}+2(\eta_{1,2}-\eta_{2,2})\overline{w}^{2m+1}.\end{aligned} \tag{49}$$

Now we shall move to the computation of the terms $\eta_{i,j}$ which are defined by

$$\eta_{i,j}=\alpha_i(m-1)\fint_{\mathbb{T}}\frac{(b_j-b_i\overline{\tau})(\alpha_j-\alpha_i\overline{\tau}^{m-1})}{(b_j-b_i\tau)^2}\overline{\tau}^m d\tau+b_i\fint_{\mathbb{T}}\frac{(b_j-b_i\overline{\tau})(\alpha_j-\alpha_i\overline{\tau}^{m-1})^2}{(b_j-b_i\tau)^3}d\tau.$$

We readily get by invoking residue theorem at $\infty$

$$\begin{aligned}\eta_{1,2} &= \alpha_1(m-1)\fint_{\mathbb{T}}\frac{(b-\overline{\tau})(\alpha_2-\alpha_1\overline{\tau}^{m-1})}{(b-\tau)^2}\overline{\tau}^m d\tau+\fint_{\mathbb{T}}\frac{(b-\overline{\tau})(\alpha_2-\alpha_1\overline{\tau}^{m-1})^2}{(b-\tau)^3}d\tau\\ &= 0.\end{aligned}$$

Always with the notation (32) we can write

$$\eta_{2,1} = (m-1)\fint_{\mathbb{T}}\frac{(1-b\overline{\tau})(\alpha_1-\overline{\tau}^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau+b\fint_{\mathbb{T}}\frac{(1-b\overline{\tau})(\alpha_1-\overline{\tau}^{m-1})^2}{(1-b\tau)^3}d\tau.$$

Developing the numerator and using the identity (41) lead to

$$\begin{aligned}\fint_{\mathbb{T}}\frac{(1-b\overline{\tau})(\alpha_1-\overline{\tau}^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau &= \fint_{\mathbb{T}}\frac{\alpha_1\overline{\tau}^m-\overline{\tau}^{2m-1}-b\alpha_1\overline{\tau}^{m+1}+b\overline{\tau}^{2m})}{(1-b\tau)^2}d\tau\\ &= \varepsilon m b^m-(2m-1)b^{2m-2}-\varepsilon(m+1)b^{m+2}+2mb^{2m}.\end{aligned}$$

Concerning the last integral term of $\eta_{2,1}$ it can be calculated as follows,

$$\begin{aligned}\fint_{\mathbb{T}}\frac{(1-b\overline{\tau})(\alpha_1-\overline{\tau}^{m-1})^2}{(1-b\tau)^3}d\tau &= \fint_{\mathbb{T}}\frac{-2\alpha_1\overline{\tau}^{m-1}+\overline{\tau}^{2m-2}-b\alpha_1^2\overline{\tau}+2\alpha_1 b\overline{\tau}^m-b\overline{\tau}^{2m-1}}{(1-b\tau)^3}d\tau\\ &= -\varepsilon m(m-1)b^{m-1}+(2m-1)(m-1)b^{2m-3}\\ &- b^3+\varepsilon m(m+1)b^{m+1}-m(2m-1)b^{2m-1}.\end{aligned}$$

Notice that we have used the following identity which can be easily derived from residue theorem,

$$\begin{aligned}\fint_{\mathbb{T}}\frac{\overline{\tau}^m}{(1-b\tau)^3}d\tau &= \fint_{\mathbb{T}}\frac{\tau^{m+1}}{(\tau-b)^3}d\tau\\ &= \frac{1}{2}(m+1)mb^{m-1}.\end{aligned}\tag{50}$$

Putting together the preceding formulae yields

$$\eta_{2,1}=\varepsilon(m+1)b^{m+2}-mb^{2m}-b^4.\tag{51}$$

As to the diagonal terms $\eta_{i,i}$, we write in view of residue theorem at $\infty$

$$\begin{aligned}\eta_{i,i} &= (m-1)\frac{\alpha_i^2}{b_i}\fint_{\mathbb{T}}\frac{(1-\overline{\tau})(1-\overline{\tau}^{m-1})}{(1-\tau)^2}\overline{\tau}^m d\tau+\frac{\alpha_i^2}{b_i}\fint_{\mathbb{T}}\frac{(1-\overline{\tau})(1-\overline{\tau}^{m-1})^2}{(1-\tau)^3}d\tau\\ &= 0.\end{aligned}\tag{52}$$

Combining (49), (50), (51) and (52) we come to the following expressions

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}(I(\phi_1(t,w))_{|t=0} &= (m-1)\big(b^2-1\big)\overline{w}+\Big(b^4+mb^{2m}-\varepsilon(m+1)b^{m+2}\Big)\overline{w}^{2m+1}\\ \frac{1}{2}\frac{d^2}{dt^2}(I(\phi_2(t,w))_{|t=0} &= (m-1)\big(\varepsilon b^{m-1}-b^{-1}\big)\overline{w}.\end{aligned}\tag{53}$$

Now plugging (44),(45) and (53) into (38) we deduce that

$$\begin{aligned}\frac{d^2}{dt^2}G_1(\lambda_m,tv_m)_{|t=0} &= 2m\Big(b^4-2\varepsilon b^{m+2}+b^{2m}\Big)e_{2m}\\ \frac{d^2}{dt^2}G_2(\lambda_m,tv_m)_{|t=0} &= 0,\end{aligned}$$

with the usual notation $e_m\triangleq \mathrm{Im}\big(\overline{w}^m\big)$. This completes the proof of the identity (35).

4.3. **Hessian computation.** The main object is to prove the points (2), (3) and (4) of Proposition 3. This will be done separately in several sections and we shall start with the second point.

4.3.1. *Computation of* $\partial_{\lambda\lambda}F_2(\lambda_m,0)$. According to Proposition 2 one has the formula

$$\begin{aligned}\partial_{\lambda\lambda}F_2(\lambda_m,0) &= 2Q\partial_\lambda\partial_f G(\lambda_m,0)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m\\ &= -2Q\partial_\lambda\partial_f G(\lambda_m,0)\Big[\partial_f G(\lambda_m,0)\Big]^{-1}\partial_\lambda\partial_f G(\lambda_m,0)v_m.\end{aligned}$$

We point out that we have used in the last line the identity (28) illustrating the loss of transversality and which implies that

$$(\mathrm{Id}-Q)\partial_\lambda\partial_f G(\lambda_m,0)v_m=\partial_\lambda\partial_f G(\lambda_m,0)v_m.$$

As we are dealing with smooth functions, Fréchet derivative coincides with Gâteaux derivative and furthermore one can use Schwarz theorem combined with (12),

$$\begin{aligned}\partial_\lambda\partial_f G(\lambda_m,0)v_m &= \big\{\partial_\lambda\mathcal{L}_{\lambda,b}v_m\big\}_{|\lambda=\lambda_m}\\ &= mb\begin{pmatrix}\varepsilon\\1\end{pmatrix}e_m.\end{aligned}$$

Using (20), (21) and (25) we obtain

$$\begin{aligned}\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m &= -mb\Big[\partial_f G(\lambda_m,0)\Big]^{-1}\Big\{\begin{pmatrix}\varepsilon\\1\end{pmatrix}e_m\Big\}\\ &= mb^{1-m}\begin{pmatrix}1\\0\end{pmatrix}\overline{w}^{m-1}.\end{aligned}\tag{54}$$

By invoking (12) we can infer that

$$\begin{aligned}\partial_\lambda\partial_f G(\lambda_m,0)\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m &= mb^{1-m}\begin{pmatrix}m&0\\0&bm\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e_m\\ &= m^2b^{1-m}\begin{pmatrix}1\\0\end{pmatrix}e_m.\end{aligned}\tag{55}$$

It follows from (22) that

$$\begin{aligned}\partial_{\lambda\lambda}F_2(\lambda_m,0) &= 2m^2b^{1-m}\Big\langle\mathbb{W},\begin{pmatrix}1\\0\end{pmatrix}\Big\rangle\mathbb{W}_m\\ &= \sqrt{2}m^2b^{1-m}\mathbb{W}_m.\end{aligned}\tag{56}$$

This finishes the proof of Proposition 3-(2).

4.3.2. *Computation of* $\partial_t\partial_\lambda F_2(\lambda_m,0)$. We intend to prove that this mixed derivative vanishes on the critical point. This will be done in several steps and the first one is to use the following formula stated in Proposition 2-(3) :

$$\begin{aligned}\partial_t\partial_\lambda F_2(\lambda_m,0) &= \frac12 Q\partial_\lambda\partial_{ff}G(\lambda_m,0)[v_m,v_m]+Q\partial_{ff}G(\lambda_m,0)\big[\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m,v_m\big]\\ &+ \frac12 Q\partial_\lambda\partial_f G(\lambda_m,0)\widetilde{v}_m.\end{aligned}$$

The easiest term is the first one and direct computations yield,

$$\begin{aligned}\partial_\lambda\partial_{ff}G_j(\lambda_m,0)[v_m,v_m] &= \frac{d^2}{dt^2}\partial_\lambda G_j(\lambda,tv_m)_{|\lambda=\lambda_m,t=0}\\ &= -\frac{d^2}{dt^2}\mathrm{Im}\Big\{w\overline{\phi_j(t,w)}\phi_j'(t,w)\Big\}\\ &= 0,\end{aligned}$$

with $\phi_j$ being the function defined in (33). However, the second second term is much more complicated and one notices first that

$$Q\partial_{ff}G(\lambda_m,0)\big[\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m,v_m\big] \;=\; Q\partial_s\partial_t G\big(\lambda_m,tv_m+s\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m\big)_{|t,s=0}.$$

It suffices now to combine Lemma 2 below with (54) in order to deduce that

$$Q\partial_s\partial_t G(\lambda_m,tv_m+s\partial_\lambda\partial_g\varphi(\lambda_m,0)v_m)_{|t,s=0}=0.$$

With regard to the last term we invoke once again Lemma 2 which implies that

$$Q\partial_\lambda\partial_f G(\lambda_m,0)\widetilde{v}_m=0.$$

Finally, after gathering the preceding results we obtain the desired result, namely,

$$\partial_t\partial_\lambda F_2(\lambda_m,0)=0.$$

To end this section we shall prove the next result which has been used in the foregoing computations.

**Lemma 2.** *Let $(m,b)\in\widehat{\mathbb{S}}$, then the following holds true.*

(i) *Expression of $\widetilde{v}_m$ :*

$$\widetilde{v}_2=0 \quad and \quad \widetilde{v}_m(w)=\widehat{\beta}_m\begin{pmatrix}-1-2b^m\\ b^{2m-1}\end{pmatrix}\overline{w}^{2m-1},\qquad m\geq 3.$$

*with*

$$\widehat{\beta}_m\triangleq\frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)}.$$

*Furthermore,*

$$Q\partial_\lambda\partial_f G(\lambda_m,0)\widetilde{v}_m=0.$$

(ii) *Let $Z_m=\alpha\overline{w}^{m-1}$ and $\widehat{Z}_m=\beta\overline{w}^{m-1}$, with $\alpha,\beta\in\mathbb{R}^2$. Then*

$$Q\partial_t\partial_s G(\lambda_m,t\,Z_m+s\,\widehat{Z}_m)_{|t,s=0}=0.$$

*Proof.* (**1**) The definition of $\widetilde{v}_m$ was given in Proposition 2 and therefore by making appeal to (25) and (35) we get

$$\begin{aligned}\widetilde{v}_m &= -\Big[\partial_f G(\lambda_m,0)\Big]^{-1}\frac{d^2}{dt^2}(Id-Q)G(\lambda_m,tv_m)_{|t=0}\\ &= -\Big[\partial_f G(\lambda_m,0)\Big]^{-1}\begin{pmatrix}\widehat{\alpha}\\0\end{pmatrix}e_{2m}\\ &= -M_{2m}^{-1}\begin{pmatrix}\widehat{\alpha}\\0\end{pmatrix}\overline{w}^{2m-1}.\end{aligned}\tag{57}$$

The frequency mode $m=2$ is very typical. Indeed, we recall that $\widehat{\alpha}=2m(b^2-\varepsilon b^m)^2$ and thus we deduce from the definition of $\varepsilon$ in (17) that for $m=2$

$$\widehat{\alpha}=0$$

which in turn implies that

$$\widetilde{v}_2=0.$$

Now we move to higher frequencies $m\geq 3$ which requires more computations. From the preceding formula for $\widetilde{v}_m$ one needs to compute the inverse of the matrix $M_{2m}$. This matrix was defined in (13) and takes the form

$$M_{2m}(\lambda_m) \;=\; \begin{pmatrix}2m\lambda_m-1-2mb^2 & b^{2m+1}\\ -b^{2m} & b\big(2m\lambda_m-2m+1\big)\end{pmatrix}.$$

Note that we are in the case $b = b_m$ which was defined in (15) and for not to burden the notation we drop the subscript $m$. Also, recall that $\lambda_m = \frac{1+b^2}{2}$ and therefore using the relation (15) we find that

$$M_{2m}(\lambda_m) \quad = \quad \begin{pmatrix} 1+2b^m & b^{2m+1} \\ -b^{2m} & -b-2b^{m+1} \end{pmatrix}.$$

Its inverse can be computed in usual way and it comes that

$$M_{2m}^{-1}(\lambda_m) \quad = \quad \frac{1}{\det\, M_{2m}(\lambda_m)} \begin{pmatrix} -b-2b^{m+1} & -b^{2m+1} \\ b^{2m} & 1+2b^m \end{pmatrix}.$$

If we insert this expression in (57) we easily obtain

$$\widetilde{v}_m \quad = \quad -\frac{b\widehat{\alpha}}{\det M_{2m}} \left( \begin{array}{c} -1-2b^m \\ b^{2m-1} \end{array} \right) \overline{w}^{2m-1}.$$

As to the determinant of $M_{2m}$ it can be factorized in the form,

$$\det\, M_{2m} = b(b^m+1)^2(b^{2m}-2b^m-1).$$

It follows from th expression of $\widetilde{\alpha}$ stated in (35) that

$$\widetilde{v}_m = \widehat{\beta}_m \left( \begin{array}{c} -1-2b^m \\ b^{2m-1} \end{array} \right) \overline{w}^{2m-1}, \quad \widehat{\beta}_m = \frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)}, \quad m \geq 3. \tag{58}$$

Now applying (12) we find the following structure

$$\partial_\lambda \partial_f G(\lambda_m, 0)\widetilde{v}_m = C_2\, e_{2m}, \quad C_2 \in \mathbb{R}^2.$$

Consequently we obtain due to (22)

$$Q\partial_\lambda \partial_f G(\lambda_m, 0)\widetilde{v}_m = 0.$$

Actually, at this stage we have not used the explicit form of $\widetilde{v}_m$. However, the exact form will be crucial in the computation of $\partial_{tt} F_2(\lambda_m, 0)$ that will be done later in Subsection 4.3.3.

(**2**) Set $Z_m = \alpha\overline{w}^{m-1}$ and $\widehat{Z}_m = \beta\overline{w}^{m-1}$ with $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$. To simplify the expressions we need to fix some notation :

$$\phi_j(t, s, w) = b_j w + (t\alpha_j + s\beta_j)\overline{w}^{m-1}. \tag{59}$$

Then from (8) and (18) we may write

$$G_j(\lambda_m, tZ_m + s\widehat{Z}_m) = \mathrm{Im}\left\{ \Big[(1-\lambda_m)\overline{\phi_j(t,s,w)} + I(\phi_j(t,s,w))\Big] w\Big(b_j + (t\alpha_j + s\beta_j)(1-m)\overline{w}^m\Big) \right\},$$

with

$$I(\phi_j(t,s,w)) = I_1(\phi_j(t,s,w)) - I_2(\phi_j(t,s,w)).$$

and

$$I_i(\phi_j(t,s,w)) = \fint_{\mathbb{T}} \frac{\overline{\phi_j(t,s,w)} - \overline{\phi_i(t,s,\tau)}}{\phi_j(t,s,w) - \phi_i(t,s,\tau)} \phi_i'(t,s,\tau)d\tau.$$

It is plain according to our notation that

$$I_i(\phi_j(t,s,w)) = \fint_{\mathbb{T}} \frac{K(\overline{\tau}, \overline{w})}{K(\tau, w)} \Big(b_i + [t\alpha_i + s\beta_i](1-m)\overline{\tau}^m\Big) d\tau,$$

with

$$K(\tau, w) \triangleq b_j w - b_i \tau + \big(t\alpha_j + s\beta_j\big)\overline{w}^{m-1} - \big(t\alpha_i + s\beta_i\big)\overline{\tau}^{m-1}.$$

Differentiating with respect to $t$ one readily obtains

$$\begin{aligned}\partial_t I_i(\phi_j(t,s,w)_{|t=s=0} &= b_i \fint_{\mathbb{T}} \frac{\alpha_j w^{m-1} - \alpha_i \tau^{m-1}}{b_j w - b_i \tau} d\tau + \alpha_i(1-m) \fint_{\mathbb{T}} \frac{(b_j \overline{w} - b_i \overline{\tau})\tau^m}{b_j w - b_i \tau} d\tau \\ &- b_i \fint_{\mathbb{T}} \frac{(b_j \overline{w} - b_i \overline{\tau})(\alpha_j \overline{w}^{m-1} - \alpha_i \overline{\tau}^{m-1})}{(b_j w - b_i \tau)^2} d\tau.\end{aligned}$$

For our purpose we do not need to compute exactly these integrals and scaling arguments appear in fact to be sufficient. Indeed, making change of variables we find real constants $a_{1,j}, b_{1,j} \in \mathbb{R}$ such that

$$\partial_t I(\phi_j(t,s,w)_{|t=s=0} = a_{1,j} w^{m-1} + b_{1,j} \overline{w}^{m+1}.$$

Similarly, differentiating with respect to the variable $s$ and repeating the same argument as before we get real constants $a_{2,j}, b_{2,j} \in \mathbb{R}$ such that

$$\partial_s I(\phi_j(t,s,w)_{|t=s=0} = a_{2,j} w^{m-1} + b_{2,j} \overline{w}^{m+1}.$$

By the same way we find real constants $a_{3,j}, b_{3,j}, \in \mathbb{R}$ with

$$\partial_t \partial_s I(\phi_j(t,s,w)_{|t=s=0} = a_{3,j} \overline{w} + b_{3,j} \overline{w}^{2m+1}.$$

Note that complete details about these computations will be given later in the proof of Proposition 5. Now it is not difficult to check that

$$\begin{aligned}\partial_t \partial_s G_j\big(\lambda_m, tZ_m + s\widehat{Z}_m\big)_{|t,s=0} &= \operatorname{Im}\bigg\{(1-m)\overline{w}^{m-1}\beta_j\, \partial_t I(\phi_j(t,s,w)_{|t=s=0} \\ &+ (1-m)\overline{w}^{m-1}\alpha_j\, \partial_s I(\phi_j(t,s,w)_{|t=s=0} \\ &+ b_j w \partial_t \partial_s I(\phi_j(t,s,w)_{|t=s=0}\bigg\}.\end{aligned}$$

Consequently we find a vector $C_3 \in \mathbb{R}^2$ such that

$$\partial_t \partial_s G\big(\lambda_m, tZ_m + s\widehat{Z}_m\big)_{|t,s=0} = C_3\, e_{2m}.$$

Hence we deduce from (22) that

$$Q \partial_t \partial_s G\big(\lambda_m, tZ_m + s\widehat{Z}_m\big)_{|t,s=0} = 0.$$

The proof of Lemma 2 is now complete. □

4.3.3. *Computation of* $\partial_{tt} F_2(\lambda_m, 0)$. We want to establish the result of Proposition 3-(4) which is the last point to check. First recall from Proposition 2 that

$$\partial_{tt} F_2(\lambda_m, 0) = \frac{1}{3} \frac{d^3}{dt^3} QG(\lambda_m, tv_m)_{|t=0} + Q\partial_{ff} G(\lambda_m, 0)\big[v_m, \widetilde{v}_m\big]$$

where the explicit expression of $\widetilde{v}_m$ is given in Lemma 2. For $m=2$ the vector $\widetilde{v}_2$ vanishes and because the second term of the right-hand side is bilinear then necessary it will be zero which implies

$$\partial_{tt} F_2(\lambda_2, 0) = \frac{1}{3} \frac{d^3}{dt^3} QG(\lambda_2, tv_2)_{|t=0}.$$

Making appeal to Proposition 4 stated later in the Subsection 4.3.4 and using the convention (17) we readily get

$$\partial_{tt} F_2(\lambda_2, 0) = -\frac{(1-b^2)^2}{b} \sqrt{2}\, \mathbb{W}_2.$$

However, for $m\geq 3$ we combine Proposition 4 with Proposition 5 and (17) leading to

$$\partial_{tt}F_2(\lambda_m,0) \;=\; \frac{1}{\sqrt{2}}\Big[m(m-1)\frac{(1-b^2)^2}{b}+\widehat{\beta}_m\widehat{\gamma}_m\Big]\mathbb{W}_m,$$

with

$$\widehat{\beta}_m=\frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)}$$

and

$$\widehat{\gamma}_m=(m-2)b+mb^{2m-1}+(4m-6)b^{m+1}+4(m-1)b^{2m+1}+2mb^{3m-1}.$$

The only point remaining concerns the formulae for $\frac{d^3}{dt^3}QG(\lambda_2,tv_2)_{|t=0}$ and $Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]$ which will be done separately in the next two subsections.

4.3.4. *Computation of* $\frac{d^3}{dt^3}QG(\lambda_m,tv_m)_{|t=0}$. The main result reads as follows.

**Proposition 4.** *Let $m\geq 2$, then we have*

$$\frac{1}{3}\frac{d^3}{dt^3}QG(\lambda_m,tv_m)_{|t=0}=-\varepsilon m(m-1)\frac{(1-b^2)^2}{b\sqrt{2}}\mathbb{W}_m,$$

where the definition of $\varepsilon$ was given in (17).

*Proof.* From Leibniz's rule we can easily check that

$$\frac{d^3}{dt^3}G_j(\lambda_m,tv_m)_{|t=0}=\textnormal{Im}\bigg\{b_jw\frac{d^3}{dt^3}I(\phi_j(t,\cdot))_{|t=0}+3(1-m)\alpha_j\overline{w}^{m-1}\frac{d^2}{dt^2}(I(\phi_j(t,\cdot))_{|t=0})\bigg\},\tag{60}$$

where the function $G_j$ was defined in (34). According to (49) there exist real coefficients $\eta_{i,j}$ such that

$$\begin{aligned}\frac{d^2}{dt^2}(I(\phi_1(t,\cdot))_{|t=0} &= 2(m-1)\big(b^2-1\big)\overline{w}+2(\eta_{1,1}-\eta_{2,1})\overline{w}^{2m+1},\\ \frac{d^2}{dt^2}(I(\phi_2(t,\cdot))_{|t=0} &= 2(m-1)\big(\varepsilon b^{m-1}-b^{-1}\big)\overline{w}+2(\eta_{1,2}-\eta_{2,2})\overline{w}^{2m+1}.\end{aligned}$$

Thus what is left is to compute $\frac{d^3}{dt^3}I(\phi_j)_{t=0}$. With the notation introduced in (37) we can verify that

$$\frac{d^3}{dt^3}I_i(\phi_j)_{t=0}=-6\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A^4}B(AC-b_iB)d\tau.$$

Using a scaling argument as before we find real constants $\widehat{\eta}_{ij}$ such that

$$\begin{aligned}\frac{1}{6}\frac{d^3}{dt^3}I_i(\phi_j)_{t=0} &= -\fint_{\mathbb{T}}\frac{B\overline{B}C}{A^2}d\tau+b_i\fint_{\mathbb{T}}\frac{B^2\overline{B}}{A^3}d\tau+\widehat{\eta}_{ij}\,\overline{w}^{3m+1}\\ &= (m-1)\overline{w}^{m+1}J_{i,j}+b_i\overline{w}^{m+1}K_{i,j}+\widehat{\eta}_{ij}\,\overline{w}^{3m+1}\end{aligned}$$

with

$$J_{i,j}=\alpha_i\fint_{\mathbb{T}}\frac{(\alpha_j-\alpha_i\overline{\tau}^{m-1})(\alpha_j-\alpha_i\tau^{m-1})\overline{\tau}^m}{(b_j-b_i\tau)^2}d\tau$$

and

$$K_{i,j}=\fint_{\mathbb{T}}\frac{(\alpha_j-\alpha_i\overline{\tau}^{m-1})^2(\alpha_j-\alpha_i\tau^{m-1})}{(b_j-b_i\tau)^3}d\tau.$$

It is now a relatively simple matter to compute $J_{i,j}$. For the term $J_{1,2}$, it is computed by combining the notation (32) with residue theorem at $\infty$

$$
\begin{aligned}
J_{1,2} &= \varepsilon b \fint_{\mathbb{T}} \frac{(1-\varepsilon b\overline{\tau}^{m-1})(1-\varepsilon b\tau^{m-1})\overline{\tau}^m}{(b-\tau)^2} d\tau \\
&= 0. \qquad (61)
\end{aligned}
$$

With respect to $J_{2,1}$ we write by the definition

$$
\begin{aligned}
J_{2,1} &= \fint_{\mathbb{T}} \frac{1+b^2-\varepsilon b(\tau^{m-1}+\overline{\tau}^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau \\
&= \fint_{\mathbb{T}} \frac{(1+b^2)\overline{\tau}^m-\varepsilon b\overline{\tau}-\varepsilon b\overline{\tau}^{2m-1}}{(1-b\tau)^2} d\tau.
\end{aligned}
$$

Therefore we get with the help of (41)

$$
J_{2,1} = (1+b^2)mb^{m-1} - \varepsilon b - \varepsilon(2m-1)b^{2m-1}. \qquad (62)
$$

As to the diagonal terms $J_{i,i}$, one has

$$
J_{i,i} = \frac{\alpha_i^2}{b_i^2} \fint_{\mathbb{T}} \frac{(1-\overline{\tau}^{m-1})(1-\tau^{m-1})\overline{\tau}^m}{(1-\tau)^2} d\tau.
$$

Because the integrand has a holomorphic continuation to $\mathbb{C}^\star$ , then we deduce by residue theorem at $\infty$

$$
J_{i,i} = 0. \qquad (63)
$$

Now we shall move to the computation of $K_{i,j}$. Using the definition and residue theorem at $\infty$ we obtain by virtue of (50)

$$
\begin{aligned}
K_{1,2} &= \fint_{\mathbb{T}} \frac{(1-\varepsilon b\overline{\tau}^{m-1})^2(1-\varepsilon b\tau^{m-1})}{(b-\tau)^3} d\tau \\
&= \varepsilon b \fint_{\mathbb{T}} \frac{\tau^{m-1}}{(\tau-b)^3} d\tau \\
&= \frac{1}{2}\varepsilon(m-1)(m-2)b^{m-2}.
\end{aligned}
$$

In regard to $K_{2,1}$ once again we invoke residue theorem which allows to get

$$
\begin{aligned}
K_{2,1} &= \fint_{\mathbb{T}} \frac{(\varepsilon b-\overline{\tau}^{m-1})^2(\varepsilon b-\tau^{m-1})}{(1-b\tau)^3} d\tau \\
&= -(1+2b^2)\fint_{\mathbb{T}} \frac{\overline{\tau}^{m-1}}{(1-b\tau)^3} d\tau + \varepsilon b \fint_{\mathbb{T}} \frac{\overline{\tau}^{2m-2}}{(1-b\tau)^3} d\tau.
\end{aligned}
$$

It follows from (50) that

$$
K_{2,1} = (m-1)\Big[(2m-1)\varepsilon\, b^{2m-2} - mb^m - \frac{1}{2}mb^{m-2}\Big]. \qquad (64)
$$

For the term $K_{i,i}$ it suffices to use residue theorem at $z=1$ as follows,

$$\begin{aligned}
K_{i,i} &= \frac{\alpha_i^3}{b_i^3}\fint_{\mathbb{T}}\frac{(1-\overline{\tau}^{m-1})^2(1-\tau^{m-1})}{(1-\tau)^3}d\tau\\
&= \frac{\alpha_i^3}{b_i^3}\fint_{C_r}\frac{3-\tau^{m-1}}{(1-\tau)^3}d\tau\\
&= \frac{1}{2}\frac{\alpha_i^3}{b_i^3}(m-1)(m-2),
\end{aligned}$$

with $C_r$ the circle of center zero and radius $r>1$. Consequently one deduces by (32)

$$K_{1,1}=\frac{1}{2}\varepsilon b^3(m-1)(m-2);\quad K_{2,2}=\frac{1}{2}b^{-3}(m-1)(m-2).$$

Putting together the preceding identities yields successively,

$$\frac{1}{6}\frac{d^3}{dt^3}I_1(\phi_1)_{t=0}=\frac{1}{2}\varepsilon b^3(m-1)(m-2)\overline{w}^{m+1}+\widehat{\eta}_{11}\overline{w}^{3m+1},$$

$$\frac{1}{6}\frac{d^3}{dt^3}I_2(\phi_1)_{t=0}=(m-1)\Big(\frac{1}{2}mb^{m-1}-\varepsilon b\Big)\overline{w}^{m+1}+\widehat{\eta}_{21}\overline{w}^{3m+1},$$

$$\frac{1}{6}\frac{d^3}{dt^3}I_1(\phi_2)_{t=0}=\frac{1}{2}\varepsilon(m-1)(m-2)b^{m-2}\overline{w}^{m+1}+\widehat{\eta}_{12}\overline{w}^{3m+1},$$

and

$$\frac{1}{6}\frac{d^3}{dt^3}I_2(\phi_2)_{t=0}=\frac{1}{2}(m-1)(m-2)b^{-2}\overline{w}^{m+1}+\widehat{\eta}_{22}\overline{w}^{3m+1}.$$

Coming back to the definition of $I(\phi_j)$ and using the foregoing identities we may find

$$\frac{1}{6}\frac{d^3}{dt^3}I(\phi_1)_{t=0}=\frac{1}{2}(m-1)\Big[\varepsilon b^3(m-2)-mb^{m-1}+2\varepsilon b\Big]\overline{w}^{m+1}+(\widehat{\eta}_{11}-\widehat{\eta}_{21})\overline{w}^{3m+1},$$

$$\frac{1}{6}\frac{d^3}{dt^3}I(\phi_2)_{t=0}=\frac{1}{2}(m-1)(m-2)\Big(\varepsilon b^{m-2}-b^{-2}\Big)\overline{w}^{m+1}+(\widehat{\eta}_{12}-\widehat{\eta}_{22})\overline{w}^{3m+1}.$$

Inserting these identities into (60) yields

$$\begin{aligned}
\frac{d^3}{dt^3}G_1(\lambda_m,tv_m)_{|t=0} &= \operatorname{Im}\Big\{w\frac{d^3}{dt^3}I(\phi_1(t,\cdot))_{|t=0}+3(1-m)\varepsilon b\overline{w}^{m-1}\frac{d^2}{dt^2}(I(\phi_1(t,\cdot))_{|t=0})_{|t=0}\Big\}\\
&= \operatorname{Im}\Big\{3m(m-1)\Big(2\varepsilon b-b^{m-1}-\varepsilon b^3\Big)\overline{w}^m+\gamma_1\overline{w}^{3m}\Big\}\\
&= 3m(m-1)\Big(2\varepsilon b-b^{m-1}-\varepsilon b^3\Big)\,e_m+\gamma_1\,e_{3m}.
\end{aligned}$$

Likewise, we get for some real constant $\gamma_2$

$$\begin{aligned}
\frac{d^3}{dt^3}G_2(\lambda_m,tv_m)_{|t=0} &= \operatorname{Im}\Big\{bw\frac{d^3}{dt^3}I(\phi_2(t,\cdot))_{|t=0}+3(1-m)\overline{w}^{m-1}\frac{d^2}{dt^2}(I(\phi_2(t,\cdot))_{|t=0})_{|t=0}\Big\}\\
&= -3m(m-1)\Big(\varepsilon b^{m-1}-b^{-1}\Big)\,e_m+\gamma_2\,e_{3m},
\end{aligned}$$

with $\gamma_1,\gamma_2\in\mathbb{R}$. Writing them in the vectorial form

$$\frac{d^3}{dt^3}G(\lambda_m,tv_m)_{|t=0}=3m(m-1)\begin{pmatrix}2\varepsilon b-b^{m-1}-\varepsilon b^3\\ -\varepsilon b^{m-1}+b^{-1}\end{pmatrix}e_m+\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}e_{3m}.$$

By means of (22) we find

$$\begin{aligned}
\frac{1}{3}\frac{d^3}{dt^3}QG(\lambda_m,tv_m)_{|t=0} &= m(m-1)\Big\langle \begin{pmatrix} 2\varepsilon b-b^{m-1}-\varepsilon b^3\\ -\varepsilon b^{m-1}+b^{-1}\end{pmatrix},\widehat{\mathbb{W}}\Big\rangle \mathbb{W}_m\\
&= \frac{1}{\sqrt{2}}m(m-1)\varepsilon\Big(2b-b^3-b^{-1}\Big)\mathbb{W}_m\\
&= -\varepsilon\, m(m-1)\frac{(1-b^2)^2}{b\sqrt{2}}\mathbb{W}_m
\end{aligned}$$

and the proposition follows. $\square$

4.3.5. *Computation of $Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]$.* Now we intend to establish the following identity which has been used in the Subsection 4.3.3.

**Proposition 5.** *The following assertions hold true.*

(i) *Case $m=2$;*
$$Q\partial_{ff}G(\lambda_m,0)[v_2,\widetilde{v}_2]=0.$$

(ii) *Case $m\geq 3$;*
$$Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]=\frac{\widehat{\beta}_m\widehat{\gamma}_m}{\sqrt{2}}\mathbb{W}_m.$$

*with*
$$\widehat{\beta}_m=\frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)}$$

*and*
$$\widehat{\gamma}_m=(m-2)b+mb^{2m-1}+(4m-6)b^{m+1}+4(m-1)b^{2m+1}+2mb^{3m-1}.$$

*Proof.* (**1**) This follows at once from the linearity of the derivation combined with Lemma 2, (**2**) Recall from the definition that

$$Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]=Q\partial_t\partial_sG(\lambda_m,tv_m+s\widetilde{v}_m)_{|t=s=0}.$$

From the explicit value of $\widetilde{v}_m$ given in Lemma 2 we can make the split $\widetilde{v}_m=\widehat{\beta}_m\widehat{v}_m$ with

$$\widehat{\beta}_m=\frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)},\quad \widehat{v}_m\triangleq\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\overline{w}^{2m-1}=\begin{pmatrix}-1-2b^m\\ b^{2m-1}\end{pmatrix}\overline{w}^{2m-1}.$$

After factoring by $\widehat{\beta}_m$, we get

$$Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]=\widehat{\beta}_mQ\partial_t\partial_sG(\lambda_m,tv_m+s\widehat{v}_m)_{|t=s=0}.$$

Let us recall from (17) and (18) that for $m\geq 3$

$$v_m=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\overline{w}^{m-1}=\begin{pmatrix}-b\\ 1\end{pmatrix}\overline{w}^{m-1}.$$

Next we shall define the functions

$$\varphi_j(t,s,w)\triangleq b_jw+t\alpha_j\overline{w}^{m-1}+s\beta_j\overline{w}^{2m-1}\quad\text{and}\quad G_j(t,s,w)\triangleq G_j(\lambda_m,tv_m+s\widehat{v}_m).$$

Then, it is easily seen that

$$G_j(t,s,w)=\operatorname{Im}\Big\{\Big[(1-\lambda_m)\overline{\varphi}_j(t,s,w)+I(\varphi_j(t,s,w))\Big]w\Big(b_j+t(1-m)\alpha_j\overline{w}^m+s(1-2m)\beta_j\overline{w}^{2m}\Big)\Big\}.$$

Differentiating successively with respect to $t$ and $s$ according to Leibniz's rule we find

$$\begin{aligned}
\partial_t\partial_s G_j(0,0,w) &= \operatorname{Im}\Big\{(1-\lambda_m)\alpha_j\beta_j\big[(1-2m)\overline{w}^m+(1-m)w^m\big]+wb_j\partial_t\partial_s I(\varphi_j(0,0,w))\\
&+ (1-2m)\beta_j\overline{w}^{2m-1}\partial_t I(\varphi_j(0,0,w))+(1-m)\alpha_j\overline{w}^{m-1}\partial_s I(\varphi_j(0,0,w))\Big\}. \qquad (65)
\end{aligned}$$

We merely note that $I(\varphi_j)=I_1(\varphi_j)-I_2(\varphi_j)$ with

$$\begin{aligned}
I_i(\varphi_j(t,s,w)) &= \fint_{\mathbb{T}}\frac{\overline{\varphi_j(t,s,w)}-\overline{\varphi_j(t,s,\tau)}}{\varphi_j(t,s,w)-\varphi_j(t,s,\tau)}\varphi_i'(t,s,\tau)d\tau\\
&= \fint_{\mathbb{T}}\frac{\overline{A}+t\overline{B}+s\overline{C}}{A+tB+sC}\big[b_i+tD+sE\big]d\tau
\end{aligned}$$

and where we adopt the notation :

$$\begin{aligned}
A &= b_jw-b_i\tau, \quad B=\alpha_j\overline{w}^{m-1}-\alpha_i\overline{\tau}^{m-1}, \quad C=\beta_j\overline{w}^{2m-1}-\beta_i\overline{\tau}^{2m-1},\\
D &= (1-m)\alpha_i\overline{\tau}^m, \quad E=(1-2m)\beta_i\overline{\tau}^{2m}.
\end{aligned}$$

We shall start with calculating the term $\partial_t I_i(\varphi_j(0,0,w))$. Then plain computations lead to

$$\partial_t I_i(\varphi_j(0,0,w))=b_i\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A^2}d\tau+\fint_{\mathbb{T}}\frac{\overline{A}D}{A}d\tau.$$

As it was emphasized before, because we use a projection $Q$ which captures only the terms on $e_m$ then we do not need to know exactly all the terms but only some of them. This elementary fact will be often used in this part. Now by a scaling argument it comes that

$$\partial_t I_i(\varphi_j(0,0,w))=b_i\fint_{\mathbb{T}}\frac{\overline{B}}{A}d\tau+\theta_{i,j}\overline{w}^{m+1},$$

with $\theta_{i,j}\in\mathbb{R}$. Therefore coming back to the notation we may write

$$\partial_t I_i(\varphi_j(0,0,w))=b_iw^{m-1}\fint_{\mathbb{T}}\frac{\alpha_j-\alpha_i\tau^{m-1}}{b_j-b_i\tau}d\tau+\theta_{i,j}\overline{w}^{m+1}.$$

As an immediate consequence we obtain successively by means of residue theorem,

$$\partial_t I_i(\varphi_i(0,0,w))=\theta_{i,i}\overline{w}^{m+1},$$

$$\begin{aligned}
\partial_t I_1(\varphi_2(0,0,w)) &= w^{m-1}\fint_{\mathbb{T}}\frac{1+b\tau^{m-1}}{b-\tau}d\tau+\theta_{1,2}\overline{w}^{m+1}\\
&= -w^{m-1}\big[1+b^m\big]+\theta_{1,2}\overline{w}^{m+1}
\end{aligned}$$

and

$$\begin{aligned}
\partial_t I_2(\varphi_1(0,0,w)) &= bw^{m-1}\fint_{\mathbb{T}}\frac{-b-\tau^{m-1}}{1-b\tau}d\tau+\theta_{2,1}\overline{w}^{m+1}\\
&= \theta_{2,1}\overline{w}^{m+1}.
\end{aligned}$$

As concerns the term $\partial_s I_i(\varphi_j(0,0,w))$, it is easy to check by elementary calculation that

$$\partial_s I_i(\varphi_j(0,0,w))=b_i\fint_{\mathbb{T}}\frac{A\overline{C}-\overline{A}C}{A^2}d\tau+\fint_{\mathbb{T}}\frac{\overline{A}E}{A}d\tau.$$

As before, using a scaling argument one finds real coefficients $\widehat{\theta}_{i,j}\in\mathbb{R}$ such that

$$\begin{aligned}\partial_s I_i(\varphi_j(0,0,w)) &= b_i \fint_{\mathbb{T}} \frac{\overline{C}}{A} d\tau+\widehat{\theta}_{i,j}\overline{w}^{2m+1}\\ &= b_i w^{2m-1}\fint_{\mathbb{T}} \frac{\beta_j-\beta_i\tau^{2m-1}}{b_j-b_i\tau}d\tau+\widehat{\theta}_{i,j}\overline{w}^{2m+1}.\end{aligned}$$

This implies in particular the vanishing of the diagonal terms in the integral, namely,

$$\partial_s I_i(\varphi_i(0,0,w))=\widehat{\theta}_{i,i}\overline{w}^{2m+1}.$$

Applying once again residue theorem we get

$$\begin{aligned}\partial_s I_2(\varphi_1(0,0,w)) &= bw^{2m-1}\fint_{\mathbb{T}} \frac{\beta_1-\beta_2\tau^{2m-1}}{1-b\tau}d\tau+\widehat{\theta}_{2,1}\overline{w}^{2m+1}\\ &= \widehat{\theta}_{2,1}\overline{w}^{2m+1}.\end{aligned}$$

Repeating the same procedure we may write

$$\begin{aligned}\partial_s I_1(\varphi_2(0,0,w)) &= w^{2m-1}\fint_{\mathbb{T}} \frac{\beta_1\tau^{2m-1}-\beta_2}{\tau-b}d\tau+\widehat{\theta}_{1,2}\overline{w}^{2m+1}\\ &= [\beta_1 b^{2m-1}-\beta_2]w^{2m-1}+\widehat{\theta}_{1,2}\overline{w}^{2m+1}\\ &= -2b^{2m-1}\big[1+b^m\big]w^{2m-1}+\widehat{\theta}_{1,2}\overline{w}^{2m+1}.\end{aligned}$$

Now, combining the preceding identities we find

$$\begin{aligned}\partial_t I(\varphi_1(0,0,w)) &= (\theta_{1,1}-\theta_{2,1})\overline{w}^{m+1},\\ \partial_s I(\varphi_1(0,0,w)) &= \big(\widehat{\theta}_{1,1}-\widehat{\theta}_{2,1}\big)\overline{w}^{2m+1}\end{aligned}$$

and

$$\begin{aligned}\partial_t I(\varphi_2(0,0,w)) &= -(1+b^m)w^{m-1}+(\theta_{1,2}-\theta_{2,2})\overline{w}^{m+1},\\ \partial_s I(\varphi_2(0,0,w)) &= -2b^{2m-1}(1+b^m)w^{2m-1}+\big(\widehat{\theta}_{1,2}-\widehat{\theta}_{2,2}\big)\overline{w}^{2m+1}.\end{aligned}$$

The next step is to compute the mixed derivative $\partial_s\partial_t I_i(\phi_j(0,0))$ which is slightly more technical. Straightforward computations yield

$$\begin{aligned}\partial_t\partial_s I_i(\varphi_j(0,0)) &= b_i\fint_{\mathbb{T}}\frac{B\overline{C}-\overline{B}C}{A^2}d\tau-2b_i\fint_{\mathbb{T}}\frac{A\overline{C}-\overline{A}C}{A^3}Bd\tau\\ &+\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A^2}Ed\tau+\fint_{\mathbb{T}}\frac{A\overline{C}-\overline{A}C}{A^2}Dd\tau.\end{aligned}$$

Performing a scaling argument we can show the existence of real constants $\zeta_{i,j}$ such that

$$\begin{aligned}\partial_t\partial_s I_i(\varphi_j(0,0)) &= \zeta_{i,j}\overline{w}^{3m+1}-b_i\fint_{\mathbb{T}}\frac{B\overline{C}}{A^2}d\tau-b_i\fint_{\mathbb{T}}\frac{C\overline{B}}{A^2}d\tau\\ &+\fint_{\mathbb{T}}\frac{\overline{B}E}{A}d\tau+\fint_{\mathbb{T}}\frac{D\overline{C}}{A}d\tau\\ &\triangleq \zeta_{i,j}\overline{w}^{3m+1}-b_iw^{m-1}I_1^{i,j}-b_i\overline{w}^{m+1}I_2^{i,j}+\overline{w}^{m+1}I_3^{i,j}+w^{m-1}I_4^{i,j}.\end{aligned}\tag{66}$$

To evaluate $I_1^{i,j}$ we write by the definition

$$I_1^{i,j}=\fint_{\mathbb{T}}\frac{\big(\alpha_j-\alpha_i\overline{\tau}^{m-1}\big)\big(\beta_j-\beta_i\tau^{2m-1}\big)}{(b_j-b_i\tau)^2}d\tau.$$

For $i=j$, we use successively residue theorem at $\infty$ and $z=1$ leading for any $r>1$ to

$$\begin{aligned}
I_1^{i,i} &= \frac{\alpha_i\beta_i}{b_i^2}\Big[-\fint_{C_r}\frac{\tau^{2m-1}}{(1-\tau)^2}d\tau+\fint_{C_r}\frac{\tau^m}{(1-\tau)^2}d\tau\Big]\\
&= \frac{\alpha_i\beta_i}{b_i^2}\big(-2m+1+m\big)\\
&= \frac{\alpha_i\beta_i}{b_i^2}(1-m),
\end{aligned}$$

with $C_r=r\mathbb{T}$. Concerning the case $i=1,j=2$ we combine residue theorem at $\infty$ with (41)

$$\begin{aligned}
I_1^{1,2} &= \fint_{\mathbb{T}}\frac{\big(\alpha_2-\alpha_1\overline{\tau}^{m-1}\big)\big(\beta_2-\beta_1\tau^{2m-1}\big)}{(b-\tau)^2}d\tau\\
&= -\alpha_2\beta_1\fint_{\mathbb{T}}\frac{\tau^{2m-1}}{(\tau-b)^2}d\tau+\alpha_1\beta_1\fint_{\mathbb{T}}\frac{\tau^m}{(\tau-b)^2}d\tau\\
&= -\alpha_2\beta_1(2m-1)b^{2m-2}+\alpha_1\beta_1 mb^{m-1}\\
&= -\beta_1\big[(2m-1)b^{2m-2}+mb^m\big].
\end{aligned}$$

As to the case $i=2,j=1$ we use (41)

$$\begin{aligned}
I_1^{2,1} &= \fint_{\mathbb{T}}\frac{\big(\alpha_1-\alpha_2\overline{\tau}^{m-1}\big)\big(\beta_1-\beta_2\tau^{2m-1}\big)}{(1-b\tau)^2}d\tau\\
&= -\alpha_2\beta_1\fint_{\mathbb{T}}\frac{\overline{\tau}^{m-1}}{(1-b\tau)^2}d\tau\\
&= -\beta_1(m-1)b^{m-2}.
\end{aligned}$$

Now we move to the term $I_2^{i,j}$ and use first the change of variables $\tau\mapsto\overline{\tau}$

$$\begin{aligned}
I_2^{i,j} &= \fint_{\mathbb{T}}\frac{\big(\alpha_j-\alpha_i\tau^{m-1}\big)\big(\beta_j-\beta_i\overline{\tau}^{2m-1}\big)}{(b_j-b_i\tau)^2}d\tau\\
&= \fint_{\mathbb{T}}\frac{\big(\alpha_j-\alpha_i\overline{\tau}^{m-1}\big)\big(\beta_j-\beta_i\tau^{2m-1}\big)}{(b_i-b_j\tau)^2}d\tau,
\end{aligned}$$

Thus similarly to $I_1^{2,1}$ and $I_1^{1,2}$ we get

$$\begin{aligned}
I_2^{1,2} &= \fint_{\mathbb{T}}\frac{\big(\alpha_2-\alpha_1\overline{\tau}^{m-1}\big)\big(\beta_2-\beta_1\tau^{2m-1}\big)}{(1-b\tau)^2}d\tau\\
&= -\alpha_1\beta_2\fint_{\mathbb{T}}\frac{\overline{\tau}^{m-1}}{(1-b\tau)^2}d\tau\\
&= \beta_2(m-1)b^{m-1}.
\end{aligned}$$

By the same way we deduce that

$$\begin{aligned}
I_2^{2,1} &= \fint_{\mathbb{T}}\frac{\big(\alpha_1-\alpha_2\overline{\tau}^{m-1}\big)\big(\beta_1-\beta_2\tau^{2m-1}\big)}{(b-\tau)^2}d\tau\\
&= -\alpha_1\beta_2(2m-1)b^{2m-2}+\alpha_2\beta_2 mb^{m-1}\\
&= \beta_2\big[(2m-1)b^{2m-1}+mb^{m-1}\big].
\end{aligned}$$

As regards the diagonal case $i = j$, one can readily write

$$\begin{aligned} I_2^{i,i} &= I_1^{i,i} \\ &= \frac{\alpha_i \beta_i}{b_i^2}(1-m). \end{aligned}$$

Let us now move to the computation of the term $I_3^{i,j}$ which is given by

$$I_3^{i,j} = (1-2m)\beta_i \fint_{\mathbb{T}} \frac{\alpha_j \overline{\tau}^{2m} - \alpha_i \overline{\tau}^{m+1}}{b_j - b_i \tau} d\tau$$

First notice that from residue theorem

$$I_3^{i,i} = 0 = I_3^{1,2}.$$

To calculate the value of $I_3^{2,1}$ we shall use (40)

$$\begin{aligned} I_3^{2,1} &= (1-2m)\beta_2 \fint_{\mathbb{T}} \frac{\alpha_1 \overline{\tau}^{2m} - \alpha_2 \overline{\tau}^{m+1}}{1 - b\tau} d\tau \\ &= (1-2m)\beta_2 \Big[\alpha_1 b^{2m-1} - \alpha_2 b^m\Big] \\ &= (2m-1)\beta_2 \big[b^{2m} + b^m\big] \end{aligned}$$

The last term that we shall deal with is $I_4^{i,j}$ whose expression is given by

$$I_4^{i,j} = (1-m)\alpha_i \fint_{\mathbb{T}} \frac{\beta_j \overline{\tau}^m - \beta_i \tau^{m-1}}{b_j - b_i \tau} d\tau.$$

For the diagonal terms $i = j$, we get for $r > 1$ and $m \in \mathbb{N}^\star$

$$\begin{aligned} I_4^{i,i} &= (1-m)\frac{\alpha_i \beta_i}{b_i} \fint_{\mathbb{T}} \frac{\overline{\tau}^m - \tau^{m-1}}{1 - \tau} d\tau \\ &= (1-m)\frac{\alpha_i \beta_i}{b_i} \fint_{C_r} \frac{\overline{\tau}^m - \tau^{m-1}}{1 - \tau} d\tau \\ &= (1-m)\frac{\alpha_i \beta_i}{b_i} \fint_{C_r} \frac{\tau^{m-1}}{\tau - 1} d\tau \\ &= (1-m)\frac{\alpha_i \beta_i}{b_i} \cdot \end{aligned}$$

For the case $i = 1, j = 2$, we write

$$\begin{aligned} I_4^{1,2} &= (1-m)\alpha_1 \fint_{\mathbb{T}} \frac{\beta_2 \overline{\tau}^m - \beta_1 \tau^{m-1}}{b - \tau} d\tau \\ &= (1-m)\alpha_1 \beta_1 \fint_{\mathbb{T}} \frac{\tau^{m-1}}{\tau - b} d\tau \\ &= (m-1) b^m \beta_1 \end{aligned}$$

The last term $I_4^{2,1}$ can be computed with the help of (40)

$$\begin{aligned} I_4^{2,1} &= (1-m)\alpha_2 \fint_{\mathbb{T}} \frac{\beta_1 \overline{\tau}^m - \beta_2 \tau^{m-1}}{1 - b\tau} d\tau \\ &= (1-m)\alpha_2 \beta_1 \fint_{\mathbb{T}} \frac{\overline{\tau}^m}{1 - b\tau} d\tau \\ &= (1-m) b^{m-1} \beta_1. \end{aligned}$$

Now let us gather the preceding identities in order to evaluate $\partial_t\partial_s I_i(\varphi_j(0,0,w))$ which is defined in (66),

$$\begin{aligned}\partial_t\partial_s I_i(\varphi_i(0,0,w)) &= \zeta_{i,i}\,\overline{w}^{3m+1}+w^{m-1}\big(-b_iI_1^{i,i}+I_4^{i,i}\big)+\overline{w}^{m+1}\big(-b_iI_2^{i,i}+I_3^{i,i}\big)\\ &= \zeta_{i,i}\,\overline{w}^{3m+1}+\frac{\alpha_i\beta_i}{b_i}(m-1)\overline{w}^{m+1}.\end{aligned}$$

Thus we find

$$\partial_t\partial_s I_1(\varphi_1(0,0,w))=(m-1)b(1+2b^m)\overline{w}^{m+1}+\zeta_{1,1}\,\overline{w}^{3m+1}$$

and

$$\partial_t\partial_s I_2(\varphi_2(0,0,w))=b^{2m-2}(m-1)\overline{w}^{m+1}+\zeta_{2,2}\,\overline{w}^{3m+1}.$$

Furthermore, we obtain

$$\begin{aligned}\partial_t\partial_s I_1(\varphi_2(0,0,w)) &= w^{m-1}\big(-I_1^{1,2}+I_4^{1,2}\big)+\overline{w}^{m+1}\big(-I_2^{1,2}+I_3^{1,2}\big)+\zeta_{1,2}\,\overline{w}^{3m+1}\\ &= \beta_1(2m-1)(b^m+b^{2m-2})w^{m-1}+\beta_2(1-m)b^{m-1}\overline{w}^{m+1}+\zeta_{1,2}\,\overline{w}^{3m+1}\\ &= (1-2m)(1+2b^m)(b^m+b^{2m-2})w^{m-1}+b^{3m-2}(1-m)\overline{w}^{m+1}+\zeta_{1,2}\,\overline{w}^{3m+1}\end{aligned}$$

and

$$\begin{aligned}\partial_t\partial_s I_2(\varphi_1(0,0,w)) &= w^{m-1}\big(-bI_1^{2,1}+I_4^{2,1}\big)+\overline{w}^{m+1}\big(-bI_2^{2,1}+I_3^{2,1}\big)+\zeta_{2,1}\,\overline{w}^{3m+1}\\ &= \beta_2(m-1)b^m\overline{w}^{m+1}+\zeta_{2,1}\,\overline{w}^{3m+1}\\ &= (m-1)b^{3m-1}\overline{w}^{m+1}+\zeta_{2,1}\,\overline{w}^{3m+1}.\end{aligned}$$

Consequently,

$$\begin{aligned}\partial_t\partial_s I(\varphi_1(0,0,w)) &= \partial_t\partial_s I_1(\varphi_1(0,0))-\partial_t\partial_s I_2(\varphi_1(0,0))\\ &= (m-1)\big(b+2b^{m+1}-b^{3m-1}\big)\overline{w}^{m+1}+(\zeta_{1,1}-\zeta_{2,1})\overline{w}^{3m+1}\end{aligned}$$

and

$$\begin{aligned}\partial_t\partial_s I(\varphi_2(0,0,w)) &= \partial_t\partial_s I_1(\varphi_2(0,0))-\partial_t\partial_s I_2(\varphi_2(0,0))\\ &= (1-2m)(1+2b^m)(b^m+b^{2m-2})w^{m-1}+(1-m)\big(b^{3m-2}+b^{2m-2}\big)\overline{w}^{m+1}\\ &+ (\zeta_{1,2}-\zeta_{2,2})\overline{w}^{3m+1}.\end{aligned}$$

Inserting the preceding identities into (65) we find real constant $\widehat{\zeta}_1$ such that

$$\begin{aligned}\partial_t\partial_s G_1(0,0,w) &= \widehat{\beta}_m\,\mathrm{Im}\Big\{\overline{w}^m\Big((m-1)(b+2b^{m+1}-b^{3m-1})+(1-\lambda_m)(1-2m)(b+2b^{m+1})\Big)\\ &+ w^m(1-m)(1-\lambda_m)(b+2b^{m+1})\Big\}+\widehat{\zeta}_1\,e_{3m}\end{aligned}$$

Since $\lambda_m=\frac{1+b^2}{2}$ then the preceding identity can be written in the form

$$\partial_t\partial_s G_1(0,0,w)\ =\widehat{\beta}_m\Big[\big(m\tfrac{b^2-1}{2}+m-1\big)\big(b+2b^{m+1}\big)-(m-1)b^{3m-1}\Big]e_m+\widehat{\zeta}_1\,e_{3m},$$

with

$$\widehat{\beta}_m=\frac{2m(b^2+b^m)^2}{(b^m+1)^2(-b^{2m}+2b^m+1)}.$$

From the identity $m(1-b^2)=2+2b^m$ given in (15) we deduce that

$$\begin{aligned}\partial_t\partial_s G_1(0,0,w) &= \widehat{\beta}_m\Big[(b+2b^{m+1})(m-2-b^m)-(m-1)b^{3m-1}\Big]e_m+\widehat{\zeta}_1\,e_{3m}\\ &= \widehat{\beta}_m\Big[-(m-1)b^{3m-1}+(m-2)b-2b^{2m+1}+(2m-5)b^{m+1}\Big]+\widehat{\zeta}_1\,e_{3m}.\end{aligned}$$

By the same way we can easily check that

$$\begin{aligned}\partial_t\partial_s G_2(0,0,w) &= \widehat{\beta}_m\Big[(2m-1)b^{m+1}+mb^{2m-1}+(4m-2)b^{2m+1}+(3m-1)b^{3m-1}\Big]e_m\\ &+ \widehat{\zeta}_2\, e_{3m},\end{aligned}$$

with $\widehat{\zeta}_2\in\mathbb{R}$. Using (22) we obtain

$$Q\partial_{ff}G(\lambda_m,0)[v_m,\widetilde{v}_m]=\frac{\widehat{\beta}_m}{\sqrt{2}}\Big\{(m-2)b+mb^{2m-1}+(4m-6)b^{m+1}+4(m-1)b^{2m+1}+2mb^{3m-1}\Big\}\mathbb{W}_m.$$

This achieves the proof of the desired identity. $\square$

## 5. Proof of the Main theorem

This last section is devoted to the proof of our main result.

*Proof.* (**1**) As we have seen in Section 2.2 the existence of m-fold V-states for $\lambda$ close enough to $\lambda_m$ reduces to solving the bifurcation equation (27) which is a two-dimensional equation

$$F_2(\lambda,t)=0,\quad (\lambda,t)\quad \text{close to}\quad (\lambda_m,0) \tag{67}$$

Now we shall check the existence of nontrivial solutions to this equation for $m=2$ and $b\notin\big\{b_{2n},n\geq 2\big\}$. Since $F_2$ is smooth then by Taylor expansion around the point $(\lambda_2,0)$ we get

$$F_2(\lambda,t)=\frac{1}{2}(\lambda-\lambda_2)^2\partial_{\lambda\lambda}F_2(\lambda_2,0)+\frac{1}{2}t^2\partial_{tt}F_2(\lambda_2,0)+\big((\lambda-\lambda_2)^2+t^2\big)\epsilon(\lambda,t)$$

with

$$\lim_{(\lambda,t)\to(\lambda_2,0)}\epsilon(\lambda,t)=0.$$

From Proposition 3 we deduce that

$$F_2(\lambda,t)=\Big[2\frac{\sqrt{2}}{b}(\lambda-\lambda_2)^2-\frac{\sqrt{2}(1-b^2)^2}{b}t^2\Big]\,\mathbb{W}_2+\big((\lambda-\lambda_2)^2+t^2\big)\epsilon(\lambda,t).$$

Without loss of generality we can assume that $\lambda_2=0$; it suffices to make dilation and translation, different in any variable. It follows at once that $F_2$ could be written in the form

$$F_2(\lambda,t)=(\lambda^2-t^2)\,\mathbb{W}_2+\big(\lambda^2+t^2\big)\epsilon(\lambda,t).$$

Introducing the new variable $s$ such that $\lambda=ts$ this equation becomes

$$\widehat{F}_2(t,s)\triangleq(s^2-1)\,\mathbb{W}_2+\big(s^2+1\big)\epsilon(ts,t)=0.$$

It is plain that

$$\widehat{F}_2(0,1)=0.$$

Moreover, by the definition and the fact $\epsilon(0,0)=0$ we easily find

$$\begin{aligned}\partial_s\widehat{F}_2(0,1) &= \lim_{s\to 1}\frac{\widehat{F}_2(0,s)-\widehat{F}_2(0,1)}{s-1}\\ &= 2\,\mathbb{W}_2\neq 0.\end{aligned}$$

Therefore we conclude by applying the implicit function theorem that near the point $(0,1)$ the solutions of $\widehat{F}_2(t,s)=0$ can be parametrized by a simple curve of the form $t\in]-\alpha,\alpha[\mapsto(t,\mu(t))$ with $\alpha>0$ is a small real number and $\mu(0)=1$. This guarantees the existence of a curve of nontrivial solution to (67).

We merely observe that by setting $\lambda=-ts$ we can repeat the preceding arguments and show the existence of nontrivial curve of solutions to $\widehat{F}_2(t,s)=0$. Therefore the set of solutions for the equation (67) is described close to $(\lambda_2,0)$ by the union of two simple curves passing through this point. However from geometrical point of view these curves describe the same domains. Indeed, let $D=D_1\backslash D_2$ be a two-fold doubly-connected V-state whose boundary $\Gamma_1\cup\Gamma_2$ is parametrized by the conformal mappings

$$\phi_j(w)=w\Big(b_j+\sum_{n\geq 1}\frac{a_{j,n}}{w^{2n}}\Big),\quad a_{j,n}\in\mathbb{R}.$$

Now denote by $\widehat{D}=iD$ the rotation of $D$ with the angle $\frac{\pi}{2}$. Then the new domain is also a V-state with two-fold structure and its conformal parametrization, which belongs to the affine space $(b_1,b_2)\mathrm{Id}+X_2$ ( recall that the space $X_m$ was introduced in (9)), is given by

$$\widehat{\phi}_j(w)=\frac{1}{i}\phi_j(iw)=w\Big(b_j+\sum_{n\geq 1}(-1)^n\frac{a_{j,n}}{w^{2n}}\Big),\quad a_{j,n}\in\mathbb{R}.$$

Thus from analytical point of view if $(\phi_1,\phi_2)$ is a nontrivial solution of the V-state problem then $(\widehat{\phi}_1,\widehat{\phi}_2)$ is a solution too and it is different from the preceding one and rotating with the same angular velocity. This implies that in the bifurcation diagram, if $(\phi_1,\phi_2)$ belongs to a given curve of bifurcation among those constructed before (note that each curves is transcritical) then $(\widehat{\phi}_1,\widehat{\phi}_2)$ belongs necessary to the second one. Otherwise, we should get four curves of bifurcation and not two. This gives the correspondence between the bifurcating curves which in fact describe the same geometry.
Finally, we point out that $\lambda$ is parametrized as follows $\lambda(t)=t\mu(t)$ with $t\in]-\alpha,\alpha[$ and $\mu(0)=1$. Therefore the range of $\lambda$ contains strictly zero which implies that the bifurcation is transcritical.

(**2**) Let $m\geq 3$ and $b=b_m$ then using Taylor expansion around the point $(\lambda_m,0)$ combined with Proposition 3 we get

$$\begin{aligned}F_2(\lambda,t)&=\frac{1}{2}(\lambda-\lambda_m)^2\,\partial_{\lambda\lambda}F_2(\lambda_m,0)+\frac{1}{2}t^2\,\partial_{tt}F_2(\lambda_m,0)+\big((\lambda-\lambda_m)^2+t^2\big)\epsilon(\lambda,t)\\&=\frac{1}{\sqrt{2}}\Big(m^2b^{1-m}(\lambda-\lambda_m)^2+\frac{\widehat{\alpha}_m}{\sqrt{2}}t^2\Big)\mathbb{W}_m+\big((\lambda-\lambda_m)^2+t^2\big)\epsilon(\lambda,t)\end{aligned}$$

with

$$\lim_{(\lambda,t)\to(\lambda_m,0)}\epsilon(\lambda,t)=0.$$

As $\widehat{\alpha}_m>0$ then the quadratic form associated to $F_2$ is definite positive meaning that $F_2$ is strictly convex around $(\lambda_m,0)$ and therefore in this neighborhood the equation (67) has no other roots than the trivial one $(\lambda_m,0)$. Consequently we deduce that for $\lambda$ close to $\lambda_m$ there is no $m-$fold V-states which bifurcating from the annulus $\mathbb{A}_b$. This achieves the proof of the Main theorem.

□

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## References


[1] A. L. Bertozzi and A. J. Majda. *Vorticity and Incompressible Flow.* Cambridge texts in applied Mathematics, Cambridge University Press, Cambridge, (2002).

[2] J. Burbea. *Motions of vortex patches.* Lett. Math. Phys. 6 (1982), no. 1, 1–16.

[3] J. Burbea, M. Landau. *The Kelvin waves in vortex dynamics and their stability.* Journal of Computational Physics, 45(1982) 127–156.

[4] A. Castro, D. Córdoba, J. Gómez-Serrano, A. Martín Zamora. *Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations.* arXiv:1409.7040.

[5] A. Castro, D. Córdoba, J. Gómez-Serrano. *Uniformly rotating analytic global patch solutions for active scalars*, arXiv:1508.01655.

[6] C. Cerretelli, C.H.K. Williamson. *A new family of uniform vortices related to vortex configurations before Fluid merger.* J. Fluid Mech. 493 (2003) 219–229.

[7] J.-Y. Chemin. *Perfect incompressible Fluids.* Oxford University Press 1998.

[8] M. G. Crandall, P. H. Rabinowitz. *Bifurcation from simple eigenvalues.* J. of Func. Analysis 8 (1971), 321–340.

[9] G. S. Deem, N. J. Zabusky. *Vortex waves : Stationary "V-states", Interactions, Recurrence, and Breaking.* Phys. Rev. Lett. 40 (1978), no. 13, 859–862.

[10] D. G. Dritschel. *The nonlinear evolution of rotating configurations of uniform vorticity*, J. Fluid Mech. **172** (1986), 157–182.

[11] D. G. Dritschel, H. Plotka. *Quasi-geostrophic shallow-water vortex-patch equilibria and their stability.* Geophys. Astrophys. Fluid Dyn. 106 (2012), no. 6, 574–595.

[12] G. R. Flierl, L. M. Polvani. *Generalized Kirchhoff vortices.* Phys. Fluids 29 (1986), 2376–2379.

[13] L. E. Fraenkel. An introduction to maximum principles and symmetry in elliptic problems. Cambridge Tracts in Mathematics, 128. Cambridge University Press, Cambridge, 2000.

[14] Y. Guo, C. Hallstrom, and D. Spirn. *Dynamics near an unstable Kirchhoff ellipse.* Comm. Math. Phys., 245(2)297–354, 2004.

[15] Z. Hassainia, T. Hmidi. *On the $V$-states for the generalized quasi-geostrophic equations.* Comm. Math. Phys. 337 (2015), no. 1, 321–377.

[16] Z. Hassainia, T. Hmidi, F. de la Hoz. *doubly-connected V-states for the generalized surface quasi-geostrophic equations.* Preprint arXiv:1412.4587.

[17] T. Hmidi, J. Mateu, J. Verdera. *Boundary Regularity of Rotating Vortex Patches.* Arch. Ration. Mech. Anal. 209 (2013), no. 1, 171–208.

[18] T. Hmidi, J. Mateu, J. Verdera. *On rotating doubly-connected vortices.* J. Differential Equations 258 (2015), no. 4, 1395–1429.

[19] T. Hmidi, F. de la Hoz, J. Mateu, J. Verdera. *Doubly-connected V-states for the planar Euler equations.* arXiv:1409.7096.

[20] T. Hmidi. *On the trivial solutions for the rotating patch model*, to appear in J. Evol. Equ., arXiv:1409.8469.

[21] T. Hmidi, J. Mateu. *Bifurcation of rotating patches from Kirchhoff vortices*, arXiv:1508.04589.

[22] J. R. Kamm. *Shape and stability of two-dimensional uniform vorticity regions.* PhD thesis, California Institute of Technology, 1987.

[23] H. Kielhöfer. *Bifurcation Theory: An Introduction with applications to partial differential equations,* Springer ( 2011).

[24] H. Kielhöfer. *Degenerate bifurcation at simple sigenvalues and stability*, J. Functional Analysis 8 (1980) 416–441.

[25] G. Kirchhoff. *Vorlesungen uber mathematische Physik* (Leipzig, 1874).

[26] A. E. H. Love. *On the Stability of certain Vortex Motions.* Proc. London Math. Soc., 25(1)18–42, 1893.

[27] P. Luzzatto-Fegiz, C. H. K. Williamson. *Stability of elliptical vortices from "Imperfect-Velocity-Impuls" diagrams.* Theor. Comput. Fluid Dyn., 24 (2010), 1-4, 181–188.

[28] T. B. Mitchell and L. F. Rossi. *The evolution of Kirchhoff elliptic vortices.* Physics of Fluids, 20(5), 2008.

[29] E. A. II Overman. *Steady-state solutions of the Euler equations in two dimensions. II. Local analysis of limiting V-states.* SIAM J. Appl. Math. 46 (1986), no. 5, 765–800.

[30] Ch. Pommerenke. *Boundary behaviour of conformal maps.* Springer-Verlag, Berlin, (1992).

[31] P. G. Saffman. *Vortex dynamics.* Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1992.

[32] Y. Tang. *Nonlinear stability of vortex patches.* Trans. Amer. Math. Soc., 304(2)617–638, 1987.

[33] Y. H. Wan. *The stability of rotating vortex patches.* Comm. Math. Phys., 107(1) 1–20, 1986.

[34] S. E. Warschawski. *On the higher derivatives at the boundary in conformal mapping,* Trans. Amer. Math. Soc. 38 (1935), no. 2, 310–340.

[35] W. Wolibner,*Un théorème sur l'existence du mouvement plan d'un fluide parfait homogène, incompressible, pendant un temps infiniment long,* Math. Z, Vol. 37, 1933, pp. 698–627.

[36] H. M. Wu, E.A. II Overman, N. J. Zabusky. *Steady-state solutions of the Euler equations in two dimensions : rotating and translating V-states with limiting cases I. Algorithms ans results*, J. Comput. Phys. 53 (1984), 42–71.

[37] Y. Yudovich. *Nonstationary flow of an ideal incompressible liquid.* Zh. Vych. Mat., 3, (1963), 1032–1066.



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